MTH4100 Calculus I

Week 5 (Thomas' Calculus Sections 2.6 to 3.5)

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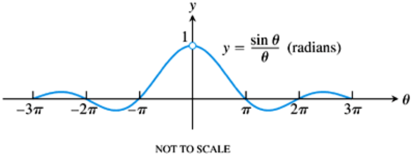
Revision of lecture 12

- a function continuous at a point
- a continuous function
  (continuous at every point of its domain)
- discontinuity at a point
  (*not necessarily* in the domain!)

Continuous extension to a point

example:

$$f(x) = \frac{\sin x}{x}$$



is defined and continuous for all $x \neq 0$. As $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it makes sense to *define a new function*

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Definition

If $\lim_{x\to c} f(x) = L$ exists, but $f(c)$ is not defined, we define a new function

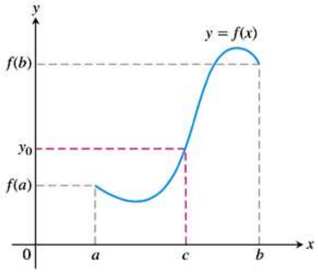
$$F(x) = \begin{cases} f(x) & \text{for } x \neq c \\ L & \text{for } x = c \end{cases},$$

which is continuous at c. It is called the continuous extension of $f(x)$ to c.

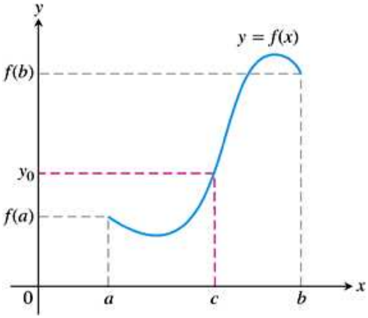
The intermediate value theorem

A function has the intermediate value property if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Geometrical interpretation of this theorem



- Any horizontal line crossing the y-axis between $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
- Continuity is essential: if f is discontinuous at any point of the interval, then the function may “jump” and miss some values.

Reading Assignment

Read

Thomas' Calculus:

page 131 / 132 about root finding

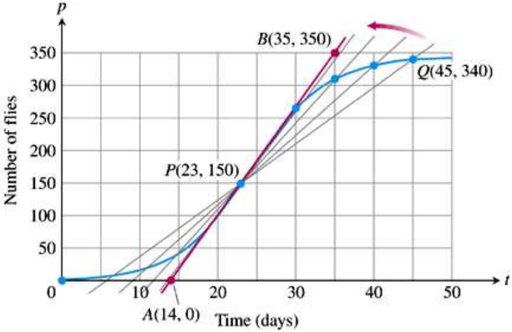
You will need a little piece of information out of this for Exercise Sheet 4!

Differentiation

Motivation: average and instantaneous rates of change

example: revisit growth of fruit fly population

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$



basic idea:

- Investigate the limit of the secant slopes as Q approaches P.
- Take it to be the slope of the tangent at P.

Now we can use **limits** to make this idea precise...

Tangent line to a parabola

example: Find the slope of the parabola $y = x^2$ at the point $P(2,4)$.

- choose a point Q a horizontal distance $h \neq 0$ away from P,

$$Q(2+h,(2+h)^2)$$

- the secant through P and Q has the slope

$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2^2}{(2+h)-2} = \frac{4+4h+h^2-4}{h} = 4+h$$

- as Q approaches P h approaches 0, hence

$$m = \lim_{h\to 0} \frac{\Delta y}{\Delta x} = \lim_{h\to 0}(4+h) = 4$$

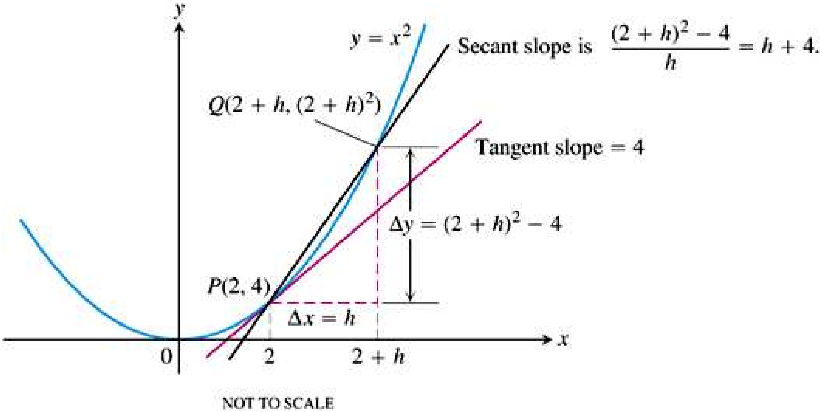
  must be the parabola's slope at P

- equation of the tangent through $P(2,4)$ is $y = y_1 + m(x - x_1)$;

$$\text{here: } y = 4 + 4(x-2) \text{ or } y = 4x - 4$$

Graphical illustration

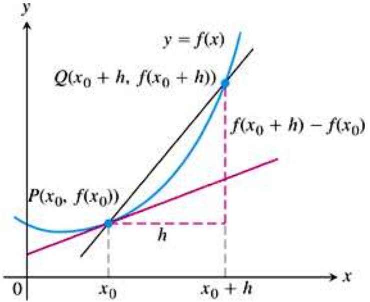
summary:



choose point Q; secant slope; tangent slope; tangent eqn.

Slope of a tangent line

now generalise to arbitrary curves and arbitrary points:



DEFINITIONS **Slope, Tangent Line**

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Recipe: calculate slope and tangent

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

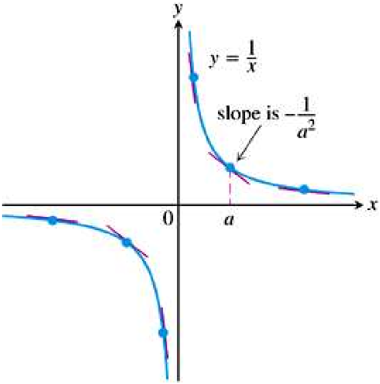
1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

Testing the recipe

example: Find slope and tangent to $y = 1/x$ at $x_0 = a \neq 0$

1. $f(a) = \dfrac{1}{a}$, $f(a+h) = \frac{1}{a+h}$
2. slope:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h\to 0} \frac{a-(a+h)}{h\cdot a(a+h)} \\ &= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2} \end{aligned}$$



2. tangent line at $(a, 1/a)$: $y = 1/a + (-1/a^2)(x-a)$ or

$$y = \frac{2}{a} - \frac{x}{a^2}$$

Difference quotient and derivative

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is called the difference quotient of f at x_0 with increment h.

The limit as h approaches 0, if it exists, is called the derivative of f at x_0.

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

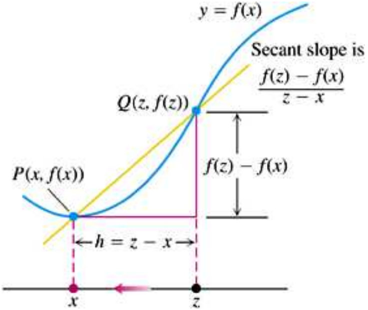
If $f'(x)$ exists, we say that f is differentiable at x.

Equivalent definition and notation

choose $z = x + h$: $h = z - x$ approaches 0 if and only if $z \to x$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



equivalent notation: if $y = f(x)$,

$$y' = f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx}$$

calculating a derivative is called

differentiation

Revision of lecture 13

- continuity:
  - continuous extension
  - intermediate value theorem
- **differentiation:**
  - tangents as limits of secants
  - definition of the derivative

Calculating derivatives from the definition

reminder:

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

example: differentiate

$$f(x) = \frac{x}{x-1}$$

$$f'(x) = \text{[calculation on whiteboard]} = -\frac{1}{(x-1)^2}$$

Calculating derivatives from the alternative definition

reminder:

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

example: differentiate

$$f(x) = \sqrt{x}$$

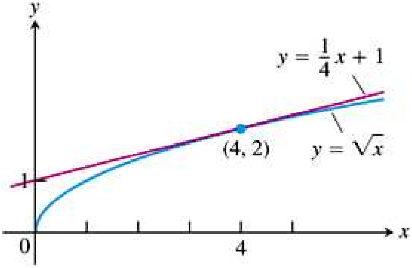
$$f'(x) = \text{[calculation on whiteboard]} = \frac{1}{2\sqrt{x}}$$

Tangent line of the square root function

$$\text{summary: } f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}}$$

calculate the *tangent line* to the curve at $x = 4$:

- $f(4) = 2$, so the line goes through the point $(4, 2)$
- slope $m = f'(4) = 1/4$
- tangent line $y = 2 + m(x - 4)$, i.e.
$$y = \frac{x}{4} + 1$$



note: one sometimes writes

$$f'(4) = \frac{d}{dx}\sqrt{x}\bigg|_{x=4} = \frac{1}{2\sqrt{x}}\bigg|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

One-sided derivatives

In analogy to one-sided limits, we define **one-sided derivatives**:

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \quad \text{right-hand derivative at } x$$

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \quad \text{left-hand derivative at } x$$

f is differentiable at x if and only if these two limits exist and are equal.

example: Show that $f(x) = |x|$ is not differentiable at $x = 0$.

- right-hand derivative at $x = 0$:

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

- left-hand derivative at $x = 0$:

$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1$$

so the right-hand and left-hand derivatives *differ*.

Differentiability implies continuity

Theorem

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof.

Trick: For $h \neq 0$, write

$$f(c+h) = f(c) + \frac{f(c+h) - f(c)}{h} h$$

By assumption, $\frac{f(c+h)-f(c)}{h} \to f'(c)$ as $h \to 0$. Therefore,

$$\lim_{h \to 0} f(c+h) = f(c) + f'(c) \cdot 0 = f(c)$$

According to definition of continuity, f is continuous at $x = c$. □

caution: the converse of the theorem is *false*!

note: theorem implies that if a function is *discontinuous* at $x = c$, then it is *not differentiable* there

Differentiation rules

(proof of one rule see ff; proof of other rules see book, Section 3.2)

Rule 1: Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0 \, .$$

Rule 2: Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1} \, .$$

Differentiation rules

Rule 3: Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}\ .$$

Proof.

$$\begin{aligned}
\frac{d}{dx}cu \quad &= \\
\text{(def. of derivative)} \quad &= \quad \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} \\
\text{(limit laws)} \quad &= \quad c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \\
(u \text{ is differentiable}) \quad &= \quad c\frac{du}{dx}
\end{aligned}$$

Differentiation rules and their application

Rule 4: Derivative Sum Rule

If u and v are differentiable functions of x, then

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} .$$

example: Differentiate $y = x^4 - 2x^2 + 2$.

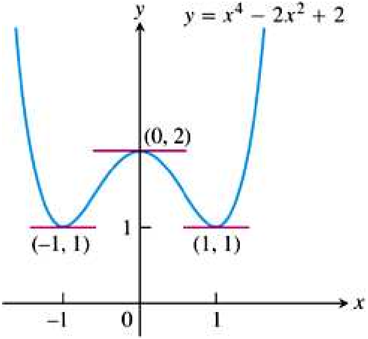
$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^4 - 2x^2 + 2) \\
\text{(rule 4)} \quad &= \frac{d}{dx}(x^4) + \frac{d}{dx}(-2x^2) + \frac{d}{dx}(2) \\
\text{(rule 3)} \quad &= \frac{d}{dx}(x^4) + (-2)\frac{d}{dx}(x^2) + \frac{d}{dx}(2) \\
\text{(rule 2)} \quad &= 4x^3 + (-2)2x + \frac{d}{dx}(2) \\
\text{(rule 1)} \quad &= 4x^3 - 4x + 0 = 4x^3 - 4x
\end{aligned}$$

Finding horizontal tangents

summary: $y = x^4 - 2x^2 + 2 \quad , \quad y' = 4x^3 - 4x$

Now find, for example, *horizontal tangents*:

$$y' = 4x^3 - 4x = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0 \quad \Rightarrow \quad x \in \{0, 1, -1\}$$



Revision of lecture 14

Differentiation:

- differentiation from first principles
- differentiable functions are continuous
- differentiation rules

Further differentiation rules

Rule 5: Derivative Product Rule

If u and v are differentiable functions of x, then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} \; .$$

Rule 6: Derivative Quotient Rule

If u and v are differentiable functions of x and $v(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} \; .$$

Common mistakes:

$$(uv)' = u'v' \quad , \quad (u/v)' = u'/v'$$

is generally **WRONG!**

Using product and quotient rules

examples: (1) Differentiate $y = (x^2 + 1)(x^3 + 3)$:

$$\text{use } \boxed{y' = (uv)' = u'v + uv'}$$

$$\text{here: } u = x^2 + 1 \,, \quad v = x^3 + 3$$

$$u' = 2x \,, \quad v' = 3x^2$$

$$y' = 2x(x^3 + 3) + (x^2 + 1)3x^2 = 5x^4 + 3x^2 + 6x$$

(2) Differentiate $y = (t^2 - 1)/(t^2 + 1)$:

$$\text{use } \boxed{y' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

$$\text{here: } u = t^2 - 1 \,, \quad v = t^2 + 1$$

$$u' = 2t \,, \quad v' = 2t$$

$$y' = \frac{2t(t^2 + 1) - (t^2 - 1)2t}{(t^2 + 1)^2} = \frac{4t}{(t^2 + 1)^2}$$

Another differentiation rule

Rule 7: Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}x^n = nx^{n-1} \ .$$

[proof: define $n = -m$ and use the quotient rule]

example:

$$\frac{d}{dx}\left(\frac{1}{x^{11}}\right) = \frac{d}{dx}(x^{-11}) = -11x^{-12} \ .$$

Higher-order derivatives

- If f' is differentiable, we call

$$f'' = (f')'$$

the **second derivative** of f.

- Notation:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y''$$

- Similarly, we write $f''' = (f'')'$ for the third derivative, and generally for the ***n*-th derivative**, $n \in \mathbb{N}_0$:

$$f^{(n)} = (f^{(n-1)})' \quad \text{with} \quad f^{(0)} = f \ .$$

Finding higher derivatives

example: Differentiate repeatedly $f(x) = x^5$ and $g(x) = x^{-2}$.

$$
\begin{aligned}
f'(x) &= 5x^4 & g'(x) &= -2x^{-3} \\
f''(x) &= 20x^3 & g''(x) &= 6x^{-4} \\
f'''(x) &= 60x^2 & g'''(x) &= -24x^{-5} \\
f^{(4)}(x) &= 120x & g^{(4)}(x) &= 120x^{-6} \\
f^{(5)}(x) &= 120 & g^{(5)}(x) &= -720x^{-7} \\
f^{(6)}(x) &= 0 & g^{(6)}(x) &= 5040x^{-8} \\
f^{(7)}(x) &= 0 & g^{(7)}(x) &= \dots
\end{aligned}
$$

Voluntary reading assignment:
Section 3.3, Practical applications of derivatives

Derivatives of trigonometric functions

(1) Differentiate $f(x) = \sin x$:

- Start with the **definition** of $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

- Use $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

- Collect terms and apply limit laws:

$$f'(x) = \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

- Use $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$ to conclude

$$f'(x) = \cos x$$

Derivatives of trigonometric functions

(2) We have just shown that $\frac{d}{dx}\sin x = \cos x$. A very similar derivation gives $\frac{d}{dx}\cos x = -\sin x$.

(3) We still need

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
\text{(quotient rule)} &= \frac{\frac{d}{dx}(\sin x)\cos x - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}
\end{aligned}$$

Summary

Derivatives of trigonometric functions

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x \\
\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\csc^2 x \\
\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x
\end{aligned}
$$

Warmup: derivative of composites

example: relating derivatives

$y = \frac{3}{2}x$ is the same as

$$y = \frac{1}{2}u \quad \text{and} \quad u = 3x \; .$$

By differentiating

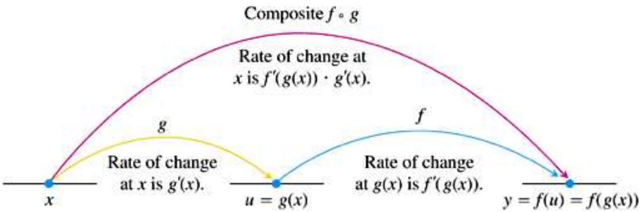
$$\frac{dy}{dx} = \frac{3}{2} \, , \quad \frac{dy}{du} = \frac{1}{2} \, , \quad \frac{du}{dx} = 3$$

we find that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \; .$$

Coincidence or general formula: *Do rates of change multiply?*

The chain rule



THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

proof: see next week

Applying the chain rule

examples: (1) Differentiate $x(t) = \cos(t^2 + 1)$.

use $\boxed{\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}}$

here: choose $x = \cos u$ and $u = t^2 + 1$ and differentiate,

$$\frac{dx}{du} = -\sin u \quad \text{and} \quad \frac{du}{dt} = 2t \ .$$

Then

$$\frac{dx}{dt} = (-\sin u)2t = \ -2t\sin(t^2 + 1) \ .$$

(2) $\frac{d}{dx} sin(x^2 + x) = \ \cos(x^2 + x)(2x + 1)$

(3) A chain with *three links*:

$$\frac{d}{dt}\tan(5 - \sin 2t) = \ \text{[Details on white board]} \ = \frac{-2\cos 2t}{\cos^2(5 - \sin 2t)} \ .$$