MTH4100 Calculus I

Week 4 (Thomas' Calculus Sections 2.4 to 2.6)

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Revision of Lecture 9

- $\epsilon - \delta$ definition of limit
- how to find δ for a given ϵ
- one-sided limits

Reminder: one-sided limits

- right-hand limit: $\lim_{x\to c^+} f(x) = L$, where $x > c$
- left-hand limit: $\lim_{x\to c^-} f(x) = M$, where $x < c$

THEOREM 6

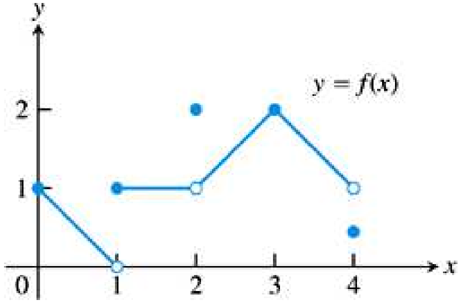
A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Limit laws, theorems for limits of polynomials and rational functions, and the sandwich theorem all hold for one-sided limits.
There is also an $\epsilon - \delta$ definition for one-sided limits (see book).

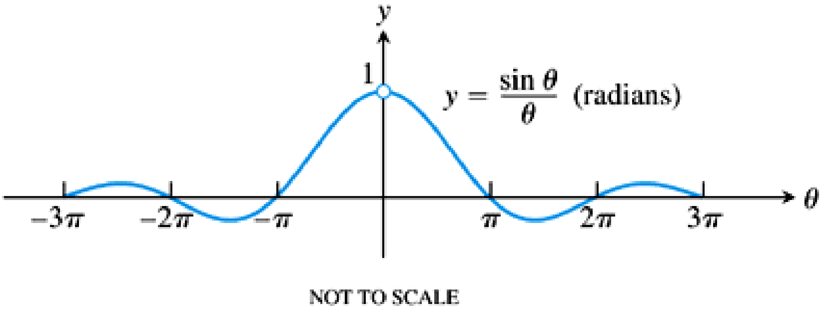
Limits of some piecewise linear function

example:



c	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$
0	n.a.	1	n.a.
1	0	1	n.a.
2	1	1	1
3	2	2	2
4	1	n.a.	n.a.

Limits involving $\sin\theta/\theta$

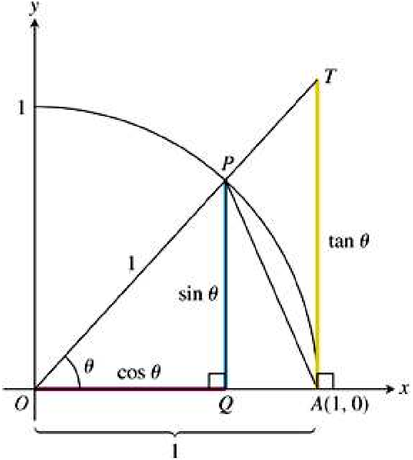


Theorem

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \quad (\theta \textit{ in radians})$$

Proof of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

show that both right-hand and left-hand limits are equal to 1:



restrict to $0 < \theta < \pi/2$

$$\sin\theta < \theta < \tan\theta$$

proof via areas of two triangles and area sector; this implies

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \quad .$$

by sandwich theorem (taking the limit as $\theta \to 0^+$)

$$1 \leq \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \leq 1 \quad .$$

symmetry: also $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$

$$\Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Applications of this theorem

examples:

(1) Compute

$$\begin{aligned}
& \lim_{h\to 0} \frac{\cos h - 1}{h} = \qquad (\sin^2(h/2) = (1-\cos h)/2) \\
= \quad & \lim_{h\to 0} \frac{1 - 2\sin^2(h/2) - 1}{h} \\
= \quad & \lim_{h\to 0} -\frac{\sin(h/2)}{h/2}\sin(h/2) \qquad (\theta = h/2) \\
= \quad & \lim_{\theta\to 0} -\frac{\sin\theta}{\theta}\sin\theta \qquad (\text{limit laws}) \\
= \quad & -1 \cdot 0 = 0
\end{aligned}$$

Applications of this theorem

(2) Compute

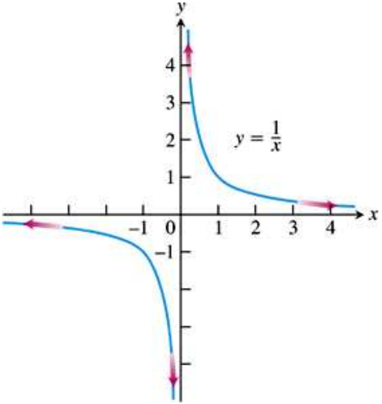
$$
\begin{aligned}
&\lim_{x\to 0} \frac{\sin 2x}{5x} = \\
= \quad &\lim_{x\to 0} \frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x} \\
= \quad &\lim_{x\to 0} \frac{2}{5}\frac{\sin 2x}{2x} \qquad (\theta = 2x) \\
= \quad &\lim_{\theta\to 0} \frac{2}{5}\frac{\sin \theta}{\theta} \qquad \text{(limit laws)} \\
= \quad &\frac{2}{5}
\end{aligned}
$$

Limits as x approaches infinity

special case of a limit:

x approaching positive/negative infinity

example:



- similar to *one-sided limit*
- use slightly modified ϵ-δ definition of a limit to capture these cases
- **idea** for this: choose a particular δ*-interval* . . .

Limits as x approaches infinity: definition

Definition

1. We say that $f(x)$ has the **limit** L **as** x **approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon \quad .$$

2. We say that $f(x)$ has the **limit** L **as** x **approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

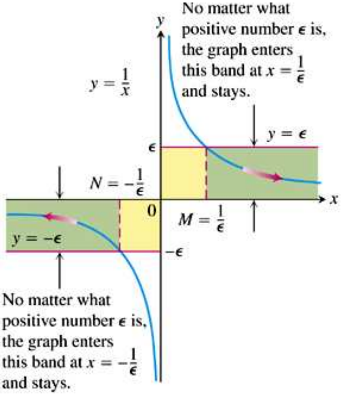
if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon \quad .$$

Limits at infinity for $f(x) = 1/x$

example:
Show that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$



Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$$

This holds if we choose $M = 1/\epsilon$ or any larger positive number.

(similarly, proof of $\lim_{x\to-\infty} \frac{1}{x} = 0$ and $\lim_{x\to\pm\infty} k = k$)

Limit laws as x approaches infinity

simply replace $x \to c$ by $x \to \pm\infty$ in the previous limit laws theorem:

Theorem

If L, M and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \text{ and } \lim_{x\to\pm\infty} g(x) = M \text{ , then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule: If s and r are integers with no common factor and $s \neq 0$, then*

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Revision of Lecture 10

- one-sided limits: example
- $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$
- limits as x approaches infinity

Calculating limits as x approaches infinity

examples: (1)

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \qquad \text{(sum rule)}$$

$$= \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = \qquad \text{(known results)}$$

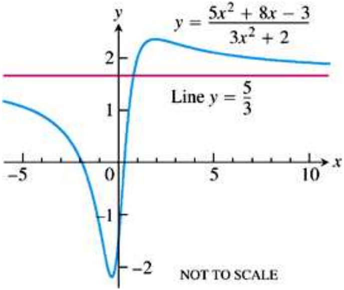
$$= 5$$

(2) method for rationals: pull out highest power of x

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} =$$

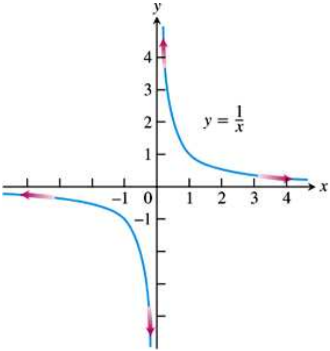
$$= \lim_{x\to\infty}\frac{x^2(5+8/x-3/x^2)}{x^2(3+2/x^2)}$$

$$= \frac{5}{3}$$



Horizontal asymptotes

example:



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

The graph approaches the line

$$y = 0$$

asymptotically: the line is an asymptote of the graph.

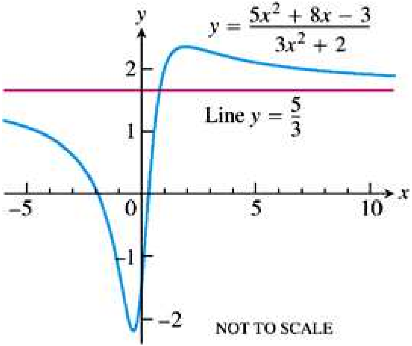
DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Calculating a horizontal asymptote

example: (already seen before)

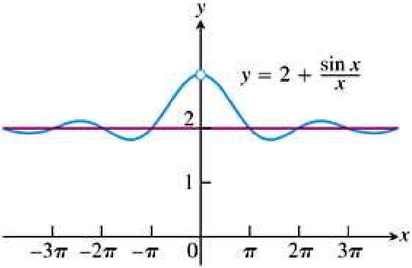


The graph has the line $y = 5/3$ as a horizontal asymptote on *both the left and the right*, because

$$\lim_{x \to \pm\infty} f(x) = \frac{5}{3} \quad .$$

Another application of the sandwich theorem...

...which also holds for limits such as $x \to \pm\infty$:
Find the horizontal asymptote of $f(x) = 2 + \frac{\sin x}{x}$.



- $0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$ (why?)
- $\lim_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0$
- therefore, by the sandwich theorem,

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

- hence,

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2$$

Oblique asymptotes

If for a rational function $f(x) = p(x)/q(x)$ the degree of $p(x)$ is *one greater* than the degree of $q(x)$, polynomial division gives

$$f(x) = ax + b + r(x) \qquad \text{with} \lim_{x\to\pm\infty} r(x) = 0$$

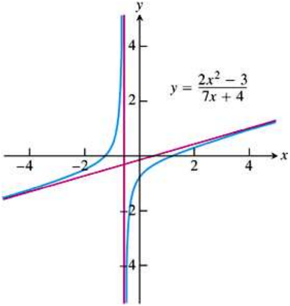
$y = ax + b$ is called an oblique (slanted) asymptote.

example: $f(x) = \dfrac{2x^2-3}{7x+4} = \dfrac{2}{7}x - \dfrac{8}{49} + \dfrac{-115}{49(7x+4)}$

$$\lim_{x\to\pm\infty} \frac{-115}{49(7x+4)} = 0, \text{ so that}$$

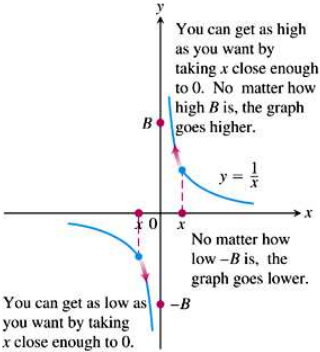
$$y = \frac{2}{7}x - \frac{8}{49}$$

is the oblique asymptote of $f(x)$.



Infinite limits

example:



$f(x) = \frac{1}{x}$ has *no limit* as $x \to 0^+$. However, it is convenient to still say that $f(x)$ *approaches* ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

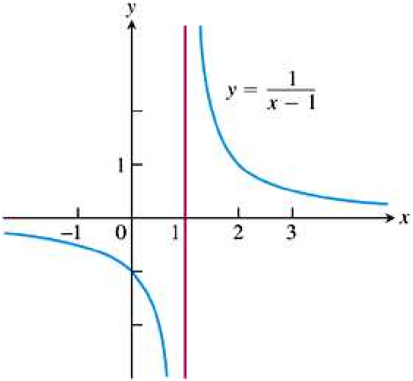
Similarly,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

note: $\lim_{x \to 0^+} \frac{1}{x} = \infty$ really means that the limit does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$!

One-sided infinite limits

example: find $\lim_{x \to 1^+} \frac{1}{x-1}$ and $\lim_{x \to 1^-} \frac{1}{x-1}$



$$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$$

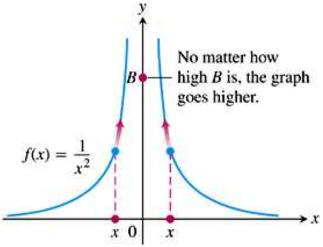
and

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

as $y = 1/(x-1)$ is just $y = 1/x$ shifted by one to the right.

Two-sided infinite limits

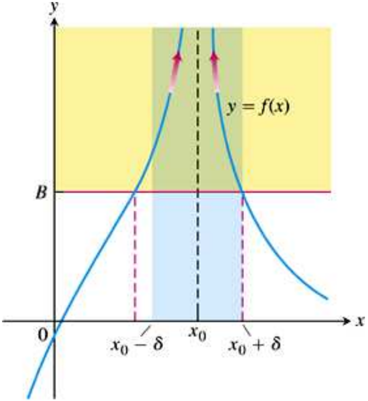
example: what is the behaviour of $f(x) = 1/x^2$ near $x = 0$?

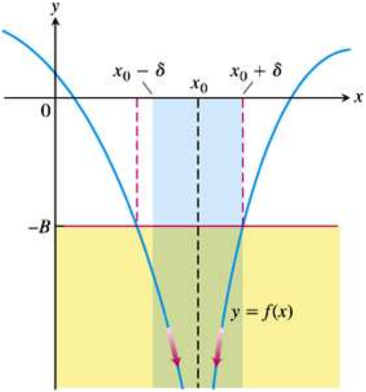


$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

as the values of $1/x^2$ are positive and become arbitrarily large as $x \to 0$.

Towards a precise definition of infinite limits





For $|x - x_0| < \delta$, the graph of $f(x)$ lies

above the line $y = B$ below the line $y = -B$

Precise definition of infinite limits

Definition

1. We say that $f(x)$ **approaches infinity as** x **approaches** x_0 and write

$$\lim_{x \longrightarrow x_0} f(x) = \infty$$

if, for every positive real number B, there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B \quad .$$

2. We say that $f(x)$ **approaches negative infinity as** x **approaches** x_0 and write

$$\lim_{x \longrightarrow x_0} f(x) = -\infty$$

if, for every negative real number $-B$, there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B \quad .$$

Using the definition

Prove that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

- given $B > 0$, find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \Rightarrow \quad \frac{1}{x^2} > B \quad ,$$

where the last inequality is equivalent to $|x| < 1/\sqrt{B}$. Therefore,

- choose $\delta = \frac{1}{\sqrt{B}}$ so that

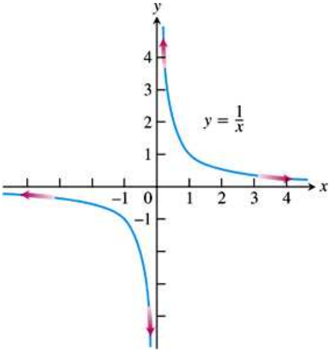
$$0 < |x| < \delta \Rightarrow \frac{1}{|x|} > \frac{1}{\delta} \Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} = B$$

- Hence, by definition

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Vertical asymptotes

example:



$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The graph approaches the line

$$x = 0$$

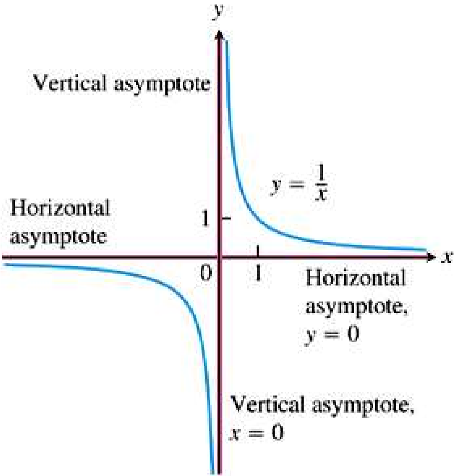
asymptotically; the line is an asymptote of the graph.

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

Summary: asymptotes for $y = 1/x$



An asymptote that is not two-sided

example: Find the horizontal and vertical asymptotes of

$$f(x) = -\frac{8}{x^2 - 4}$$

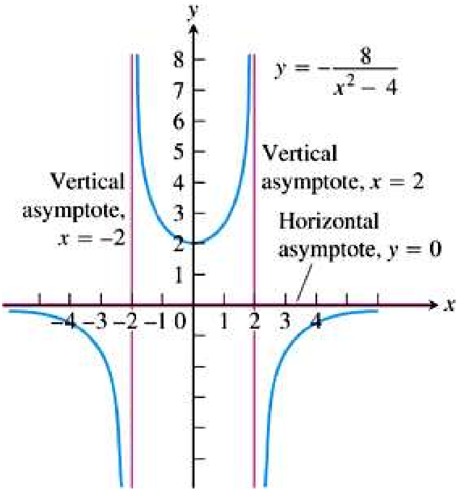
Check for the behaviour as $x \to \pm\infty$ and as $x \to \pm 2$ (why?):

- $\lim_{x\to\pm\infty} f(x) = 0$, approached from *below*.
- $\lim_{x\to -2^-} f(x) = -\infty$, $\lim_{x\to -2^+} f(x) = \infty$
- $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to 2^+} f(x) = -\infty$ (because $f(x)$ is *even*)

Asymptotes are

$$\boxed{y = 0\,, \quad x = -2\,, \quad x = 2}$$

A one-sided asymptote



curve approaches the x-axis from *only one side*

Asymptotes of another rational function

example: Find the asymptotes of

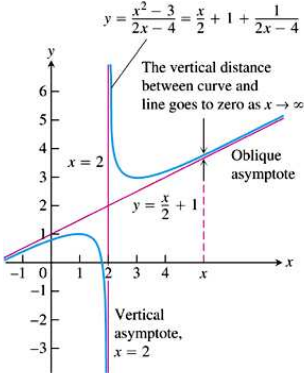
$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- Rewrite by **polynomial division:**

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

- Asymptotes are

$$\boxed{y = \tfrac{x}{2} + 1\,, \quad x = 2}$$



We say that $x/2 + 1$ dominates when x is large and that $1/(2x - 4)$ dominates when x is near 2.

Revision of Lecture 11

- horizontal asymptotes
- oblique asymptotes
- infinite limits
- vertical asymptotes

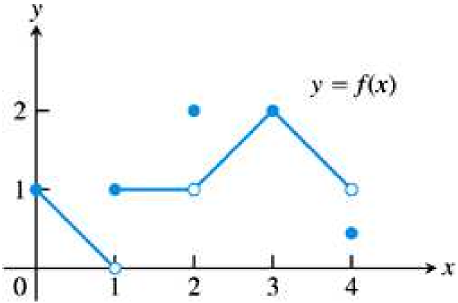
Continuity

Intuitive approach towards continuity

Definition (informal)

Any function whose graph can be sketched over its domain in one continuous motion, i.e. *without lifting the pen*, is an example of a continuous function.

example:

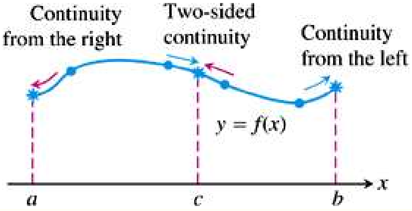


This function is continuous on $[0, 4]$ *except at* $x = 1, x = 2$ and $x = 4$.

Continuity at a point

More precisely, we need to define continuity at *interior* and at *end points*.

example:



DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

Continuity at an interior point

For any $x = c$ in the domain of f one defines:

- right-continuous: $\lim_{x \to c^+} f(x) = f(c)$
- left-continuous: $\lim_{x \to c^-} f(x) = f(c)$

A function f is *continuous at an interior point* $x = c$ if and only if it is *both right-continuous and left-continuous at* c.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions:

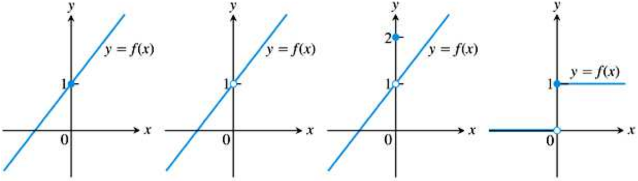
1. $f(c)$ exists.
2. f has a limit as x approaches c.
3. The limit equals the function value.

Note the difference to a function merely having a limit!

A catalogue of discontinuity types

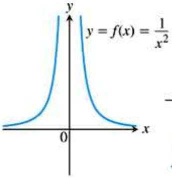
If a function f is not continuous at a point c, we say that f is discontinuous at c. Note that c need not be in the domain of f.

examples:

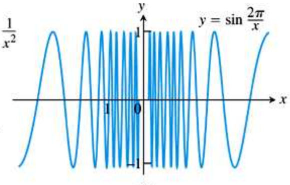


(a) (b) (c) (d)

continuous not continuous jump discontinuity



(e)



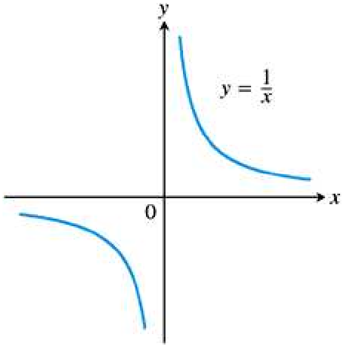
(f)

infinite discontinuity oscillating discontinuity

Continuous functions

- A function is continuous on an interval if and only if it is continuous at every point of the interval.
- A continuous function is a function that is continuous at every point of its domain.

example:



- $y = 1/x$ is a continuous function: It is continuous at every point of its domain.
- It has nevertheless a *discontinuity* at $x = 0$: No contradiction, because it is not defined there.

Algebraic combinations of continuous functions

Previous limit laws straightforwardly imply:

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

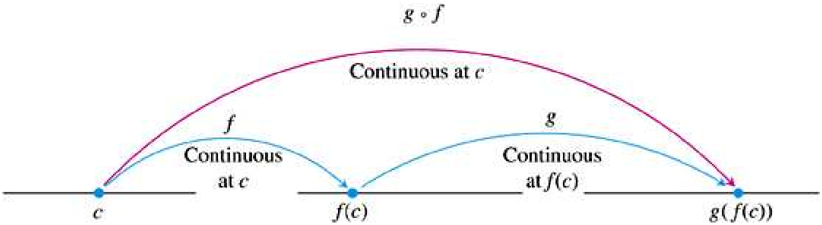
1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

example: $f(x) = x$ and constant functions are continuous $\Rightarrow$ polynomials and rational functions are also continuous

Continuity for composites

THEOREM 10 Composite of Continuous Functions

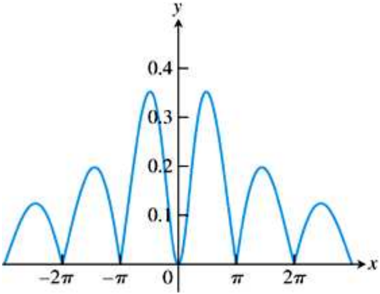
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



Applying the previous two theorems

Note that $y = \sin x$ and $y = \cos x$ are everywhere continuous:
Show that $y = \left|\frac{x \sin x}{x^2+2}\right|$ is everywhere continuous.

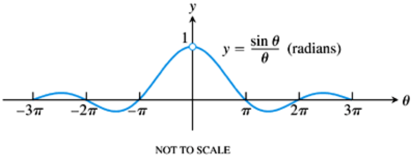
- $f(x) = \frac{x \sin x}{x^2+2}$ is continuous (why?)
- $g(x) = |x|$ is continuous (why?)
- therefore $y = g \circ f(x)$ is continuous



Continuous extension to a point

example:

$$f(x) = \frac{\sin x}{x}$$



is defined and continuous for all $x \neq 0$. As $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it makes sense to *define a new function*

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Definition

If $\lim_{x\to c} f(x) = L$ exists, but $f(c)$ is not defined, we define a new function

$$F(x) = \begin{cases} f(x) & \text{for } x \neq c \\ L & \text{for } x = c \end{cases},$$

which is continuous at c. It is called the continuous extension of $f(x)$ to c.

Finding continuous extensions

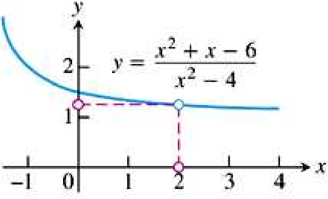
example: Find the continuous extension of $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$ to $x = 2$.

For $x \neq 2$, $f(x)$ is equal to

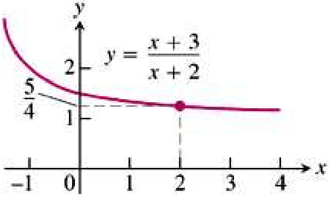
$$F(x) = \frac{x+3}{x+2} \text{ (why?)}$$

$F(x)$ is the continuous extension of $f(x)$ to $x = 2$, as

$$\lim_{x \to 2} f(x) = \frac{5}{4} = F(2)$$



(a)

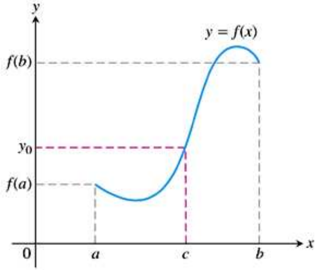


(b)

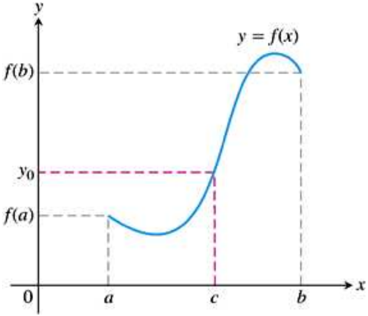
The intermediate value theorem

A function has the intermediate value property if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Geometrical interpretation of this theorem



- Any horizontal line crossing the y-axis between $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
- Continuity is essential: if f is discontinuous at any point of the interval, then the function may "jump" and miss some values.