Anomalous Diffusion, Anomalous Time Series, and the models that describe them.

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NATURAL ENVIRONMENT RESEARCH COUNCIL

Thanks

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Summary

- Why stochastic models ?
- Textbook stochastic models
- Noah, Joseph and volatility bunching
- A physics of fractals ?
- From fractals back to physics ?
- Pitfalls: 1. Walks are not noises
- 2. Memory not always from self-similarity
- 3. Choice of fractal models

WHY STOCHASTIC MODELS ?

- We need to use stochastic (or partly stochastic) models in physics and the geosciences, particularly in time series analysis.
  - Partly as models when computational bandwidth or other issues prevent a fully deterministic model
  - But also as paradigms to help us frame the right statistical questions about data.

- Motivation not only the classic problems but also increasing importance of topics like extremes and large deviations.
- Couple of examples of relevance to the environmental sciences:
  - Extreme weather events
  - Solar Terrestrial Physics ("Space Weather")

50 year storm heading for Britain bringing downpours and 70mph winds

England and Wales are back on flood alert as Britain braces for a "once in 50 years" storm which could bring a month's worth of rain in 24 hours and 70mph winds.

Image 1 of 4

Persistent heavy rain is on the way

50 year storm heading for Britain bringing downpours and 70mph winds

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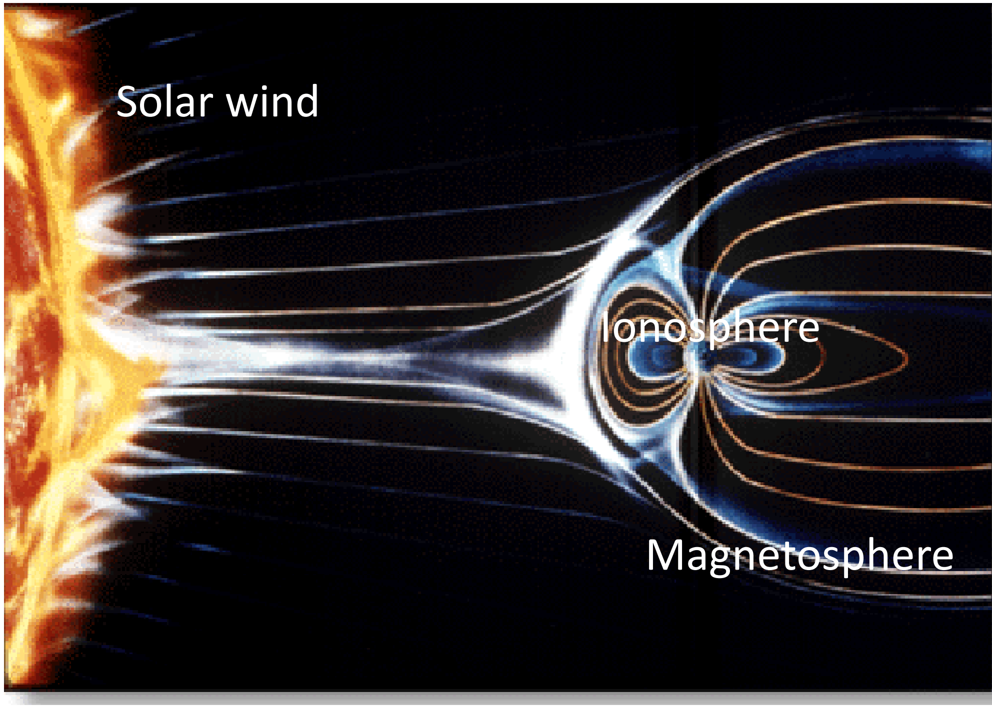
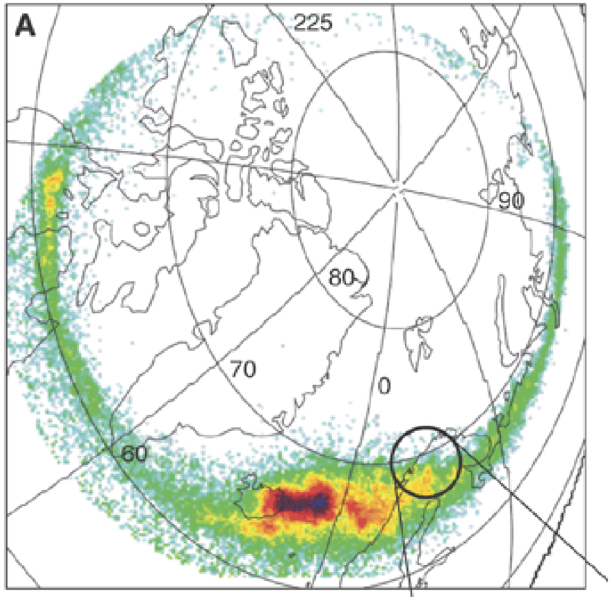
Image 1 of 4

Persistent heavy rain is on the way

Solar-Terrestrial Coupling

System

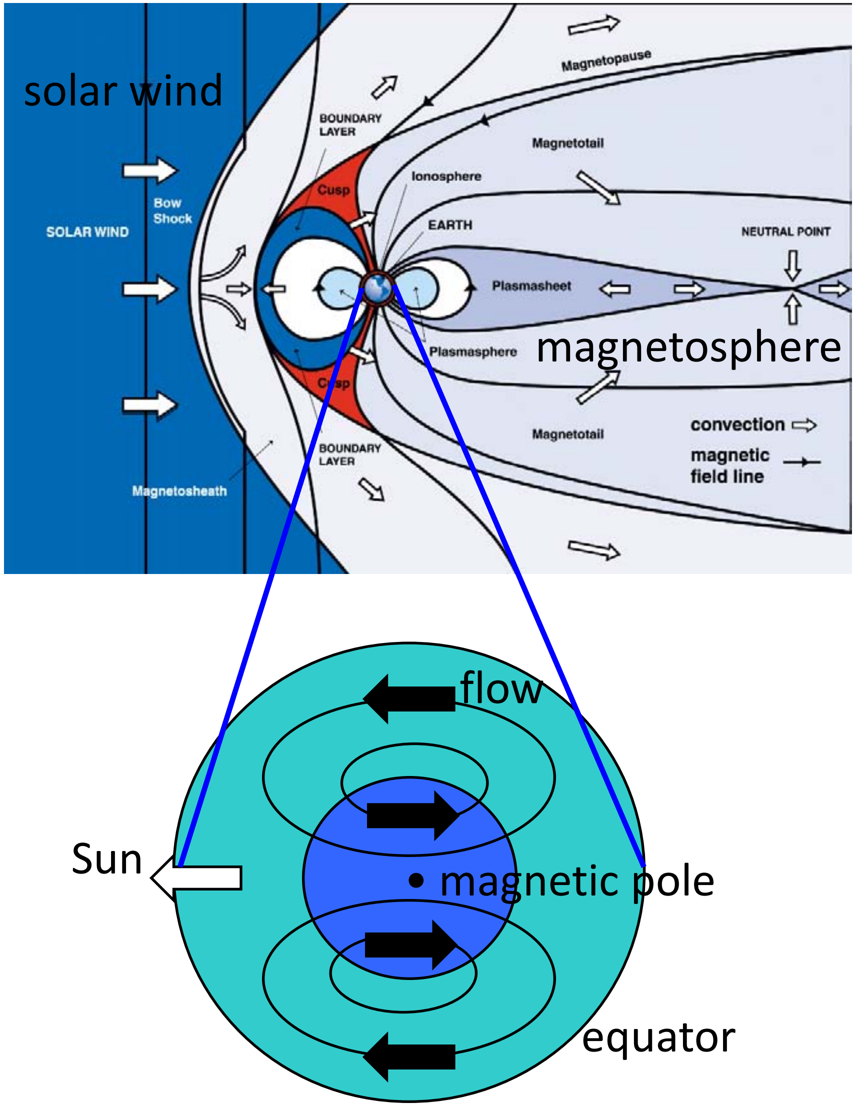




NASA Polar UVI

Space-based instrumentation

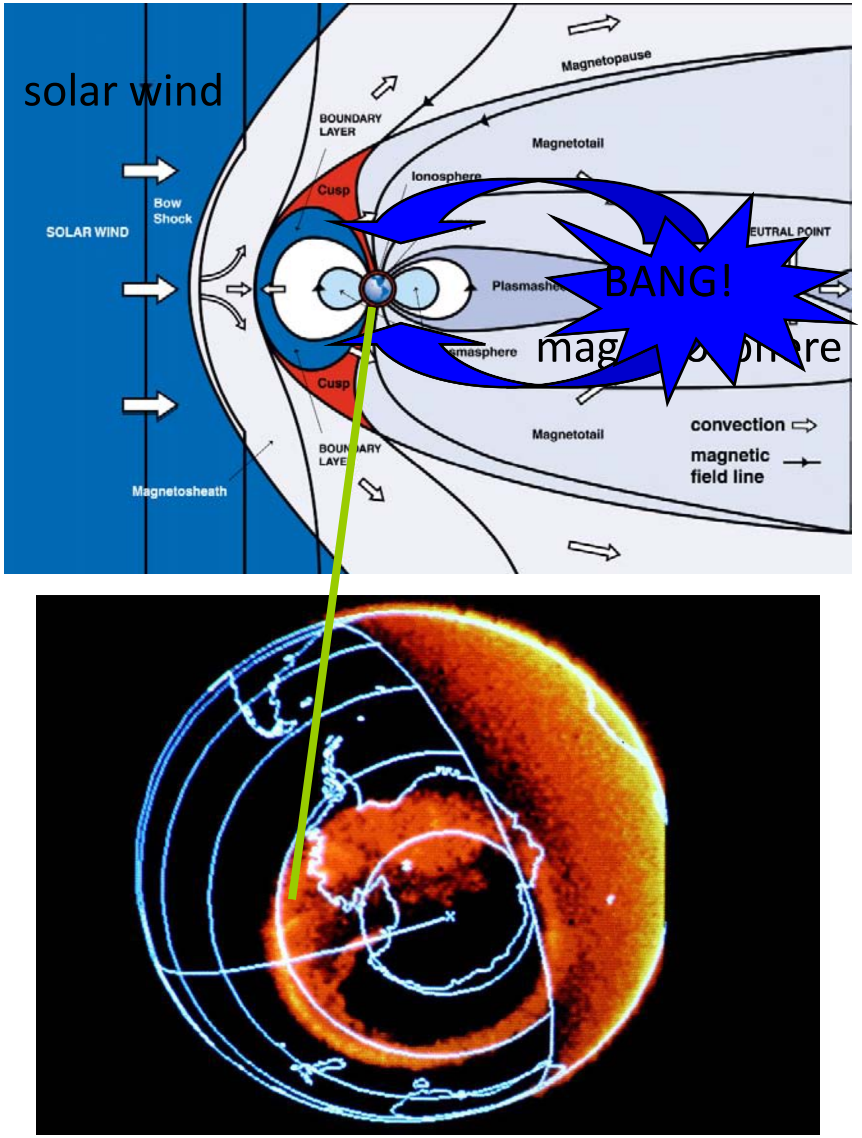
Convection (DP2)



- Mass, momentum and energy input from reconnection at solar wind - magnetosphere interface.
- Plasma circulation from day to night over poles and from night to day around flanks.

Courtesy Mervyn Freeman

Substorms (DP1)



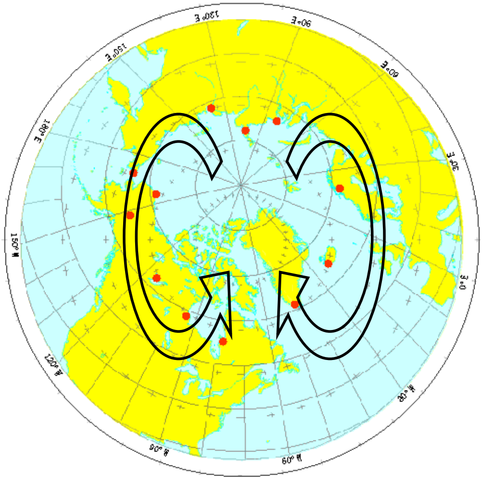
- Irregular, large-scale releases of energy in magnetotail

  -substorms.

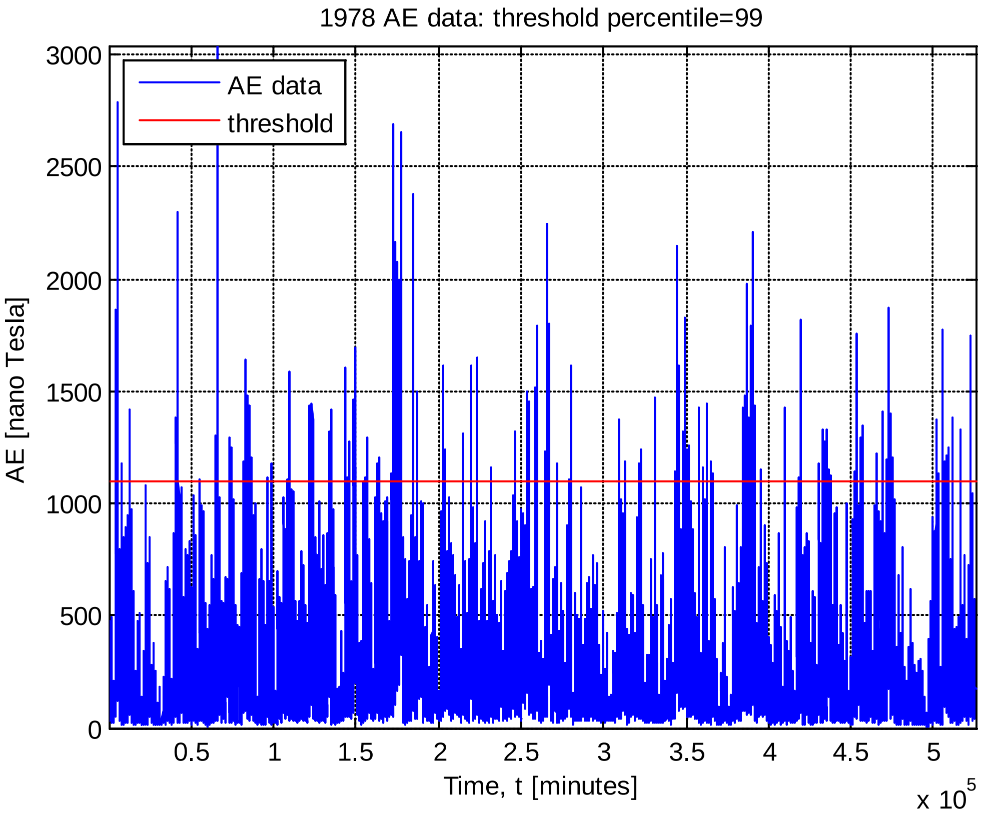
- Intense magnetic field-aligned currents accelerate particles to cause aurora.

Courtesy Mervyn Freeman

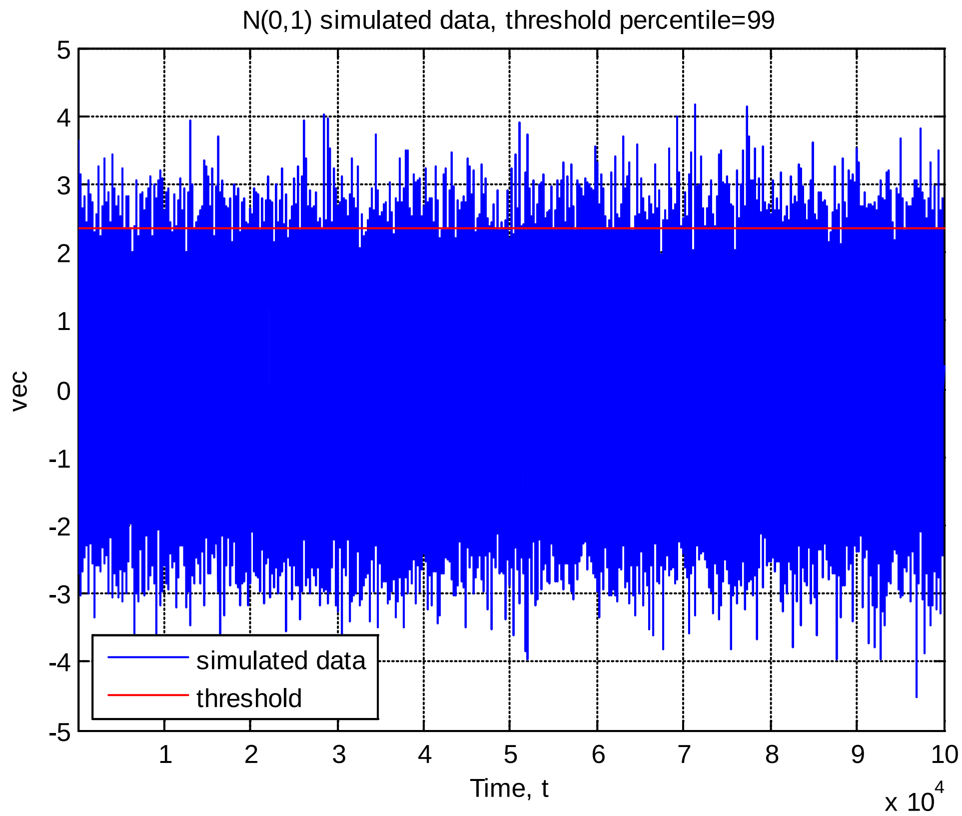
Auroral Electrojet Index



Ground-based Instrumentation



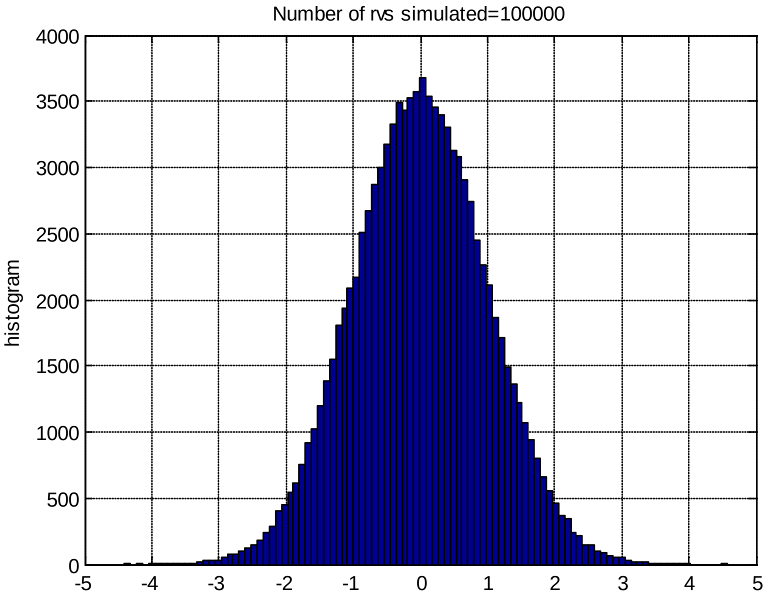
“TEXTBOOK” STOCHASTIC MODEL

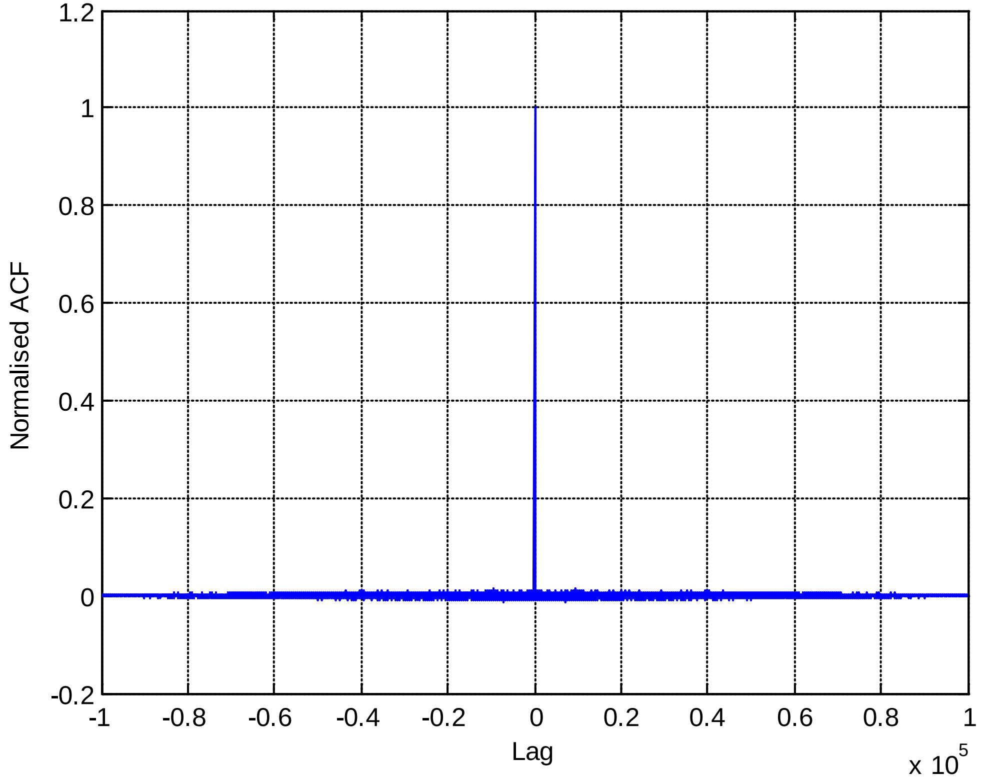


Simplest textbook stochastic time series model: independent, identically distributed (iid), Gaussian, “white”, stationary noise.

White Gaussian noise has short tailed amplitude distribution

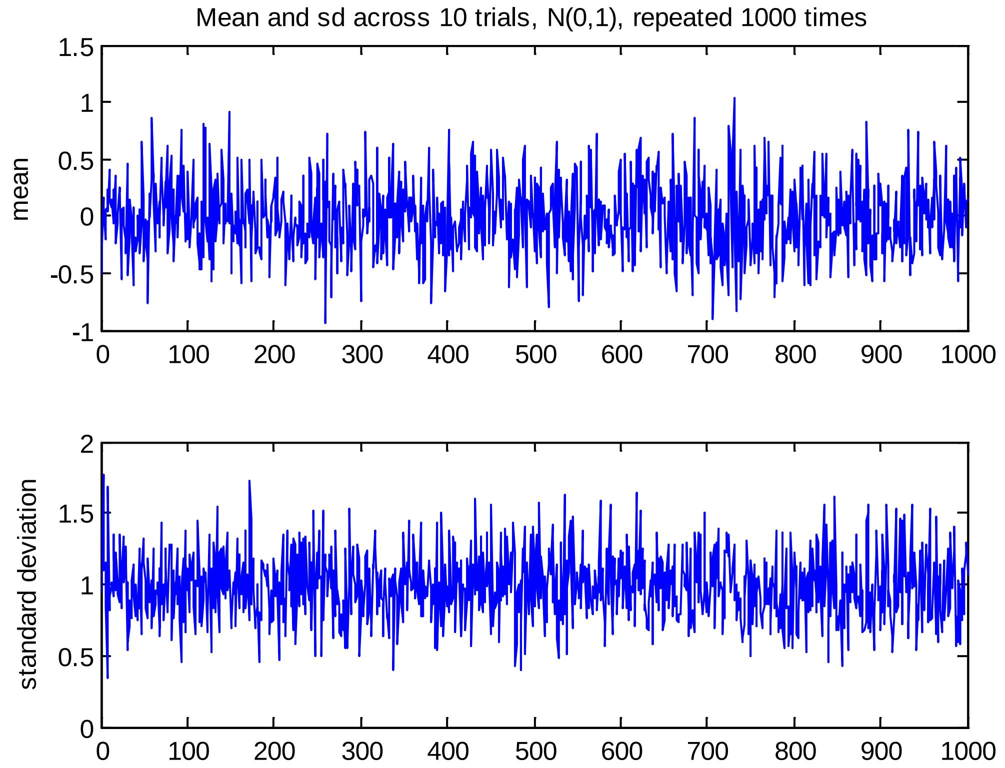
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(x-\mu)^2 / 2\sigma^2\right]$$





$$\rho(\tau) = \sum_{i=1}^{N-\tau} (X(t_i) - \mu)(X(t_i + \tau) - \mu)$$

ACF is delta-correlated in time, here plotted normalised by mean square of signal



its moments do not grow with time.

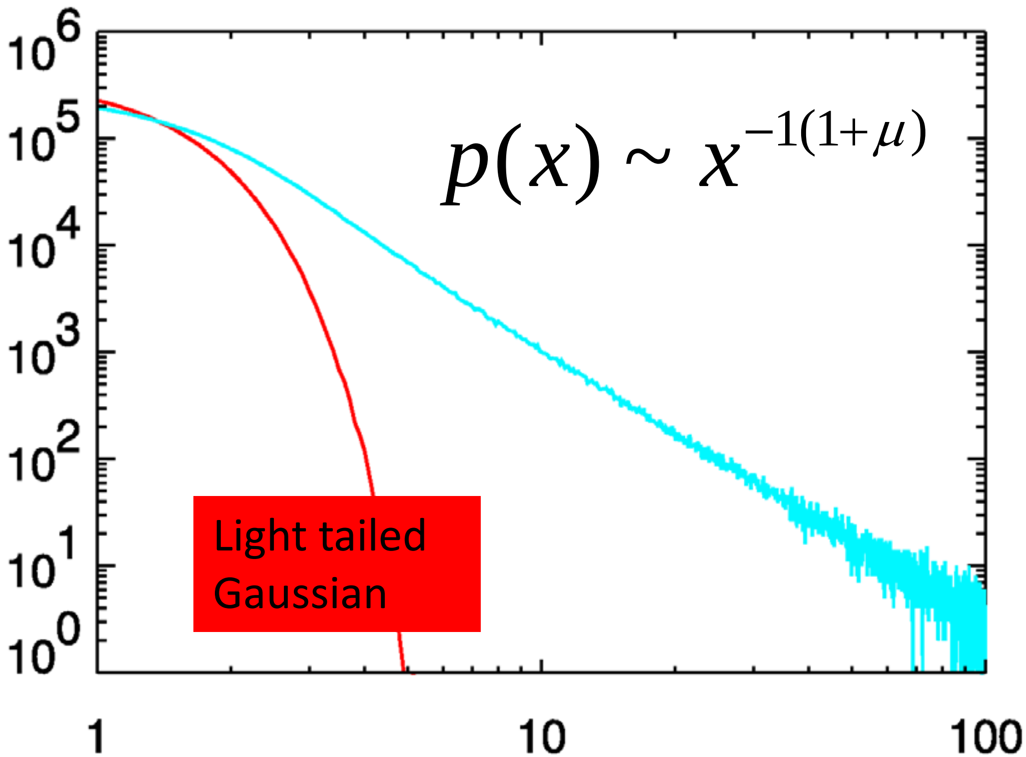
NOAH, JOSEPH AND VOLATILITY

- In stark contrast, **Mandelbrot's** classic work in the 1960s and early 1970s focused particularly on 3 “anomalous” effects seen in time series drawn from the natural and economic sciences, each of which represented a strong departure from one of the above properties of white noise.

- AE time series exhibits them **all** to some degree – has forced us to explore beyond simple noise models.

  [Have removed some unpublished AE work, paper in preparation]

Noah effect: Light and heavy tails



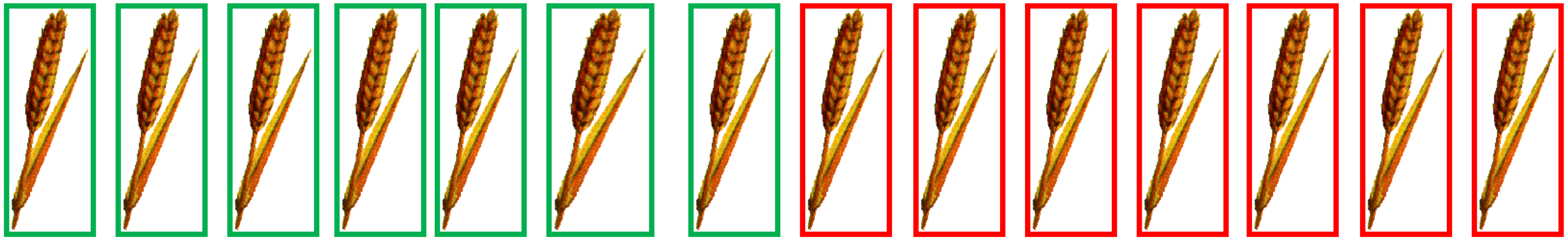
Heavy tailed power law

$$p(x) \sim \exp(-x^2 / 2\sigma^2)$$

One way to explore tail of distribution is via probability density function p(x).

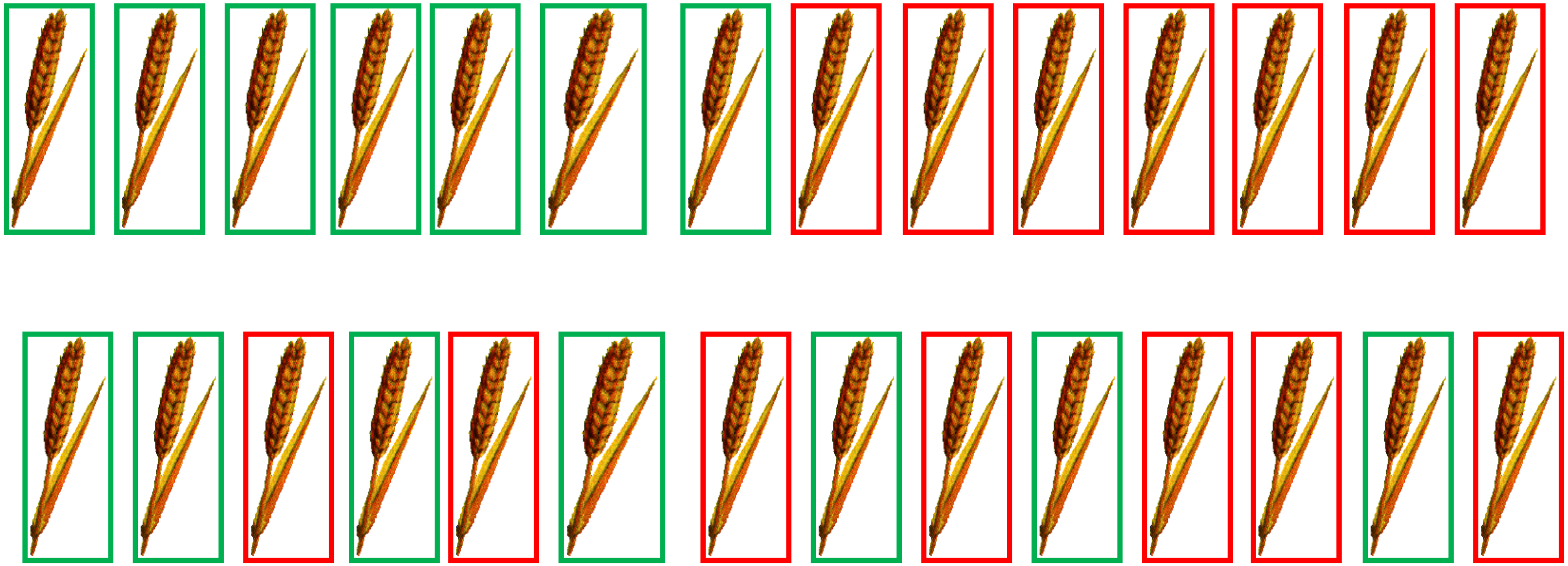
Joseph Effect

long range serial dependence in time

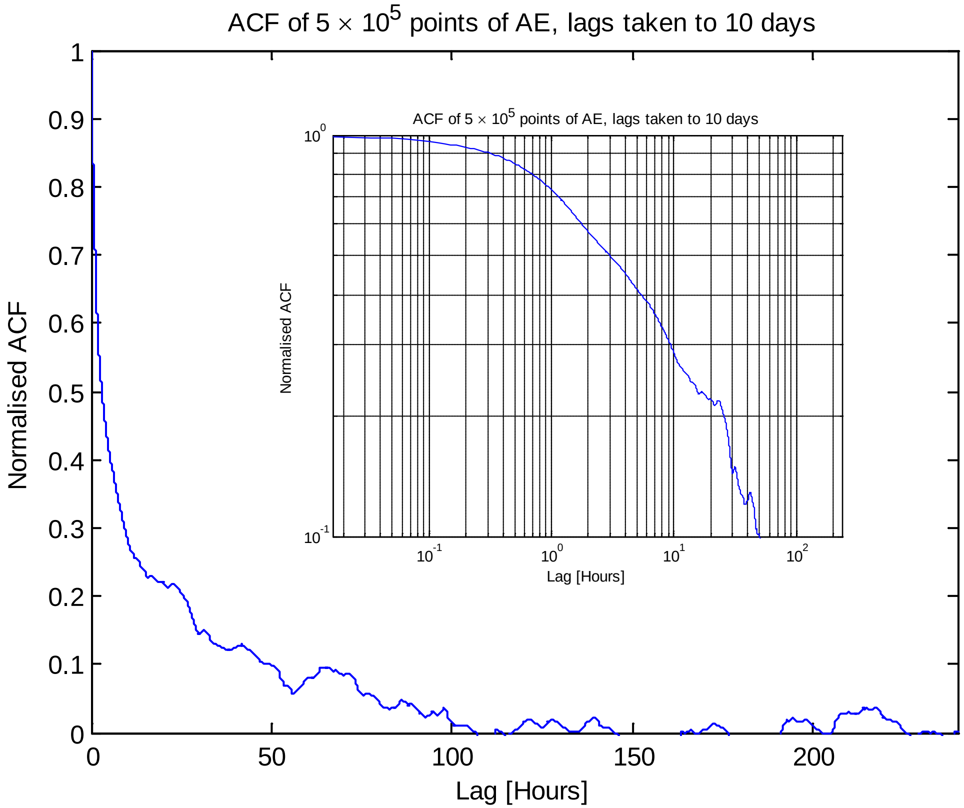


Joseph Effect

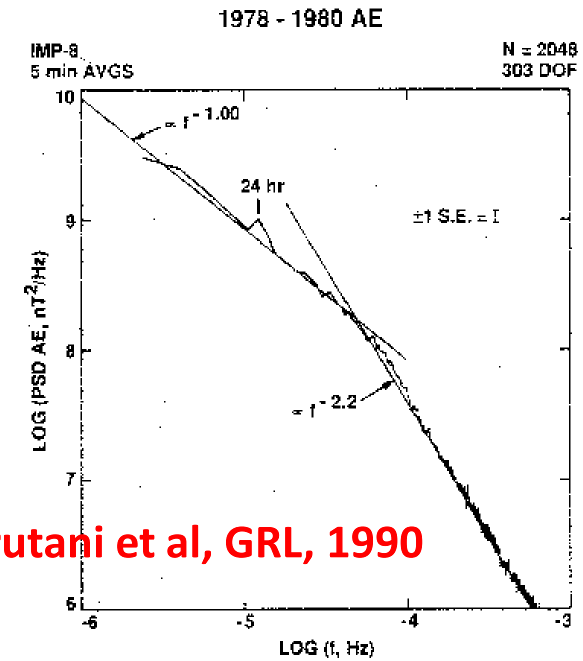
long range serial dependence in time



Long tailed ACF suggestive of LRD



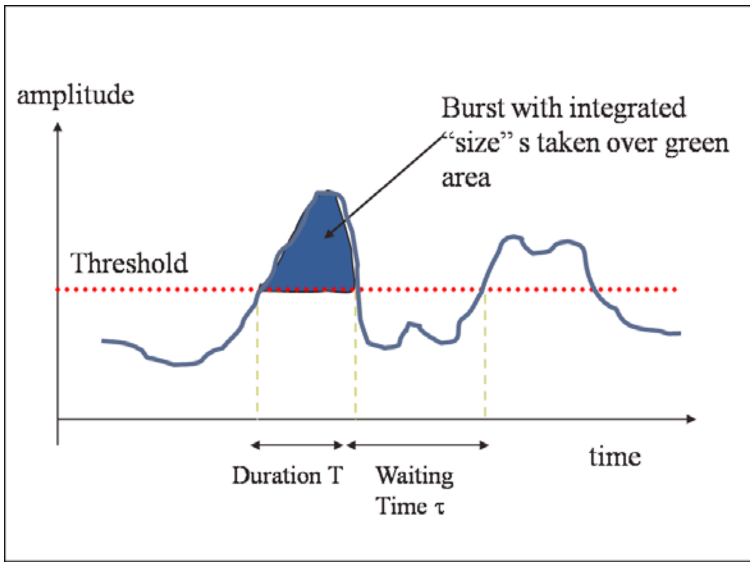
$$\rho(k) \sim c_p k^{2d-1}$$



Tsurutani et al, GRL, 1990

A singular power spectral density S(f) or an ACF that tends to infinity when summed are definitions of LRD **in a stationary time series.**

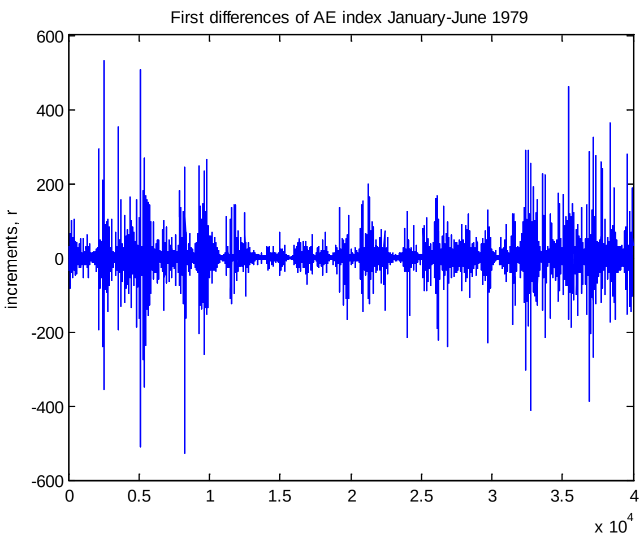
"Bursts" illustrate effect of LRD

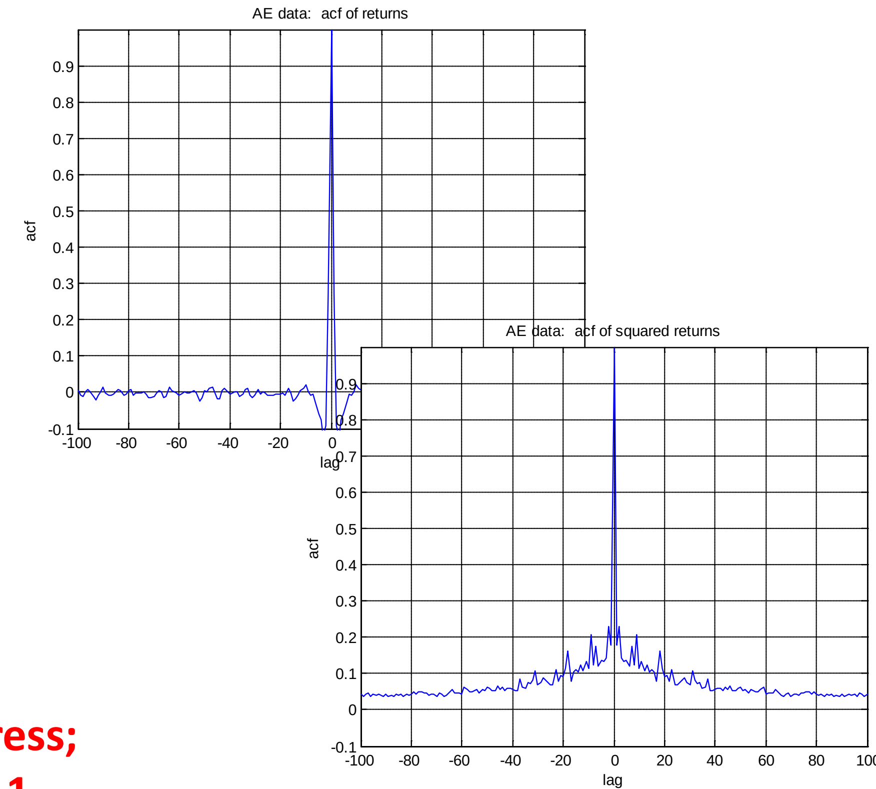


$$A_I = \int_{t_i}^{t_{i+1}} (Y(t') - \theta)\,dt'$$

Define burst area A as integrated area of time series above threshold between upward crossing and next downward crossing.

“Volatility clustering" (correlations between the absolute value of the time series- or here its first differences).





Watkins et al, AGU Monograph, in press;
See also Rypdal and Rypdal, JGR, 2011.

Common thread in all 3 effects is fractality.

Dilating power law amplitude pdf by scale factor leaves it a power law. Contrast Gaussian pdf with scale length from standard deviation.

Similarly dilating power law ACF leaves it power law. Contrast exponential ACF with scale length from e-folding time.

Can relate dilation of time series amplitude X to dilation in time via self-affinity exponent H.

$$x(\lambda \Delta t) = \lambda^{H} x(\Delta t)$$

However a single self-similarity exponent may not be enough, even in early 1970s Mandelbrot was thinking about multifractals i.e. a spectrum of exponents.

A PHYSICS OF FRACTALS ?

Bak et al's self-organised criticality, introduced **(PRL, 1987; PRE, '88)** to unify Noah & Joseph effects through self-similar "avalanches".

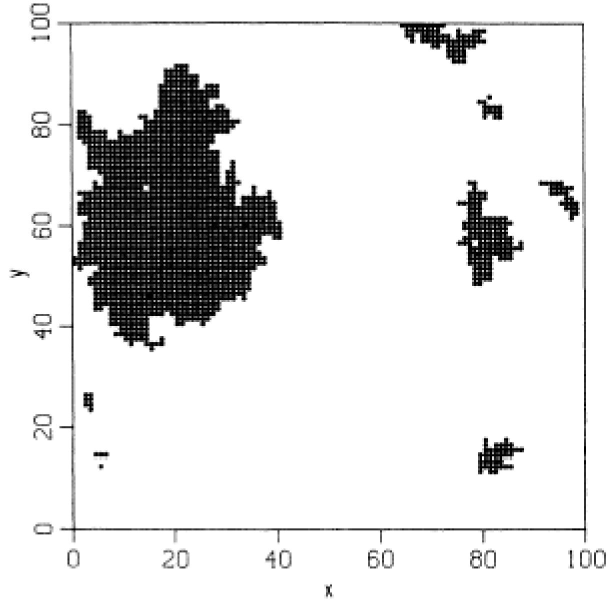
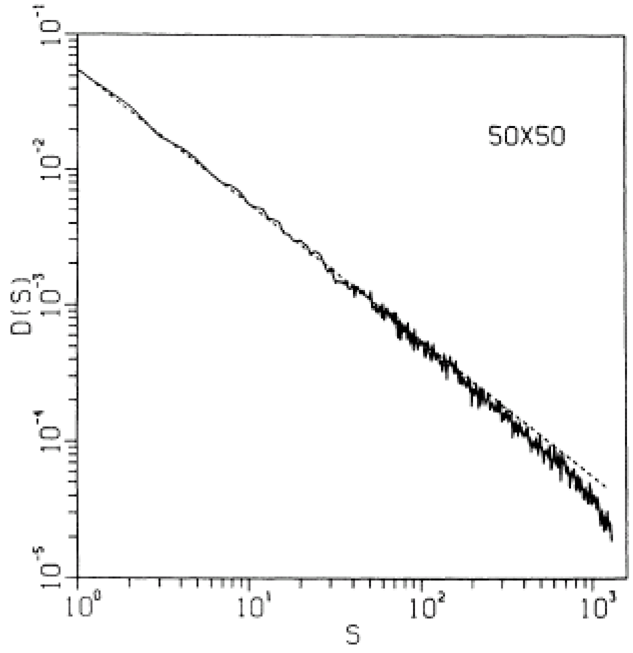
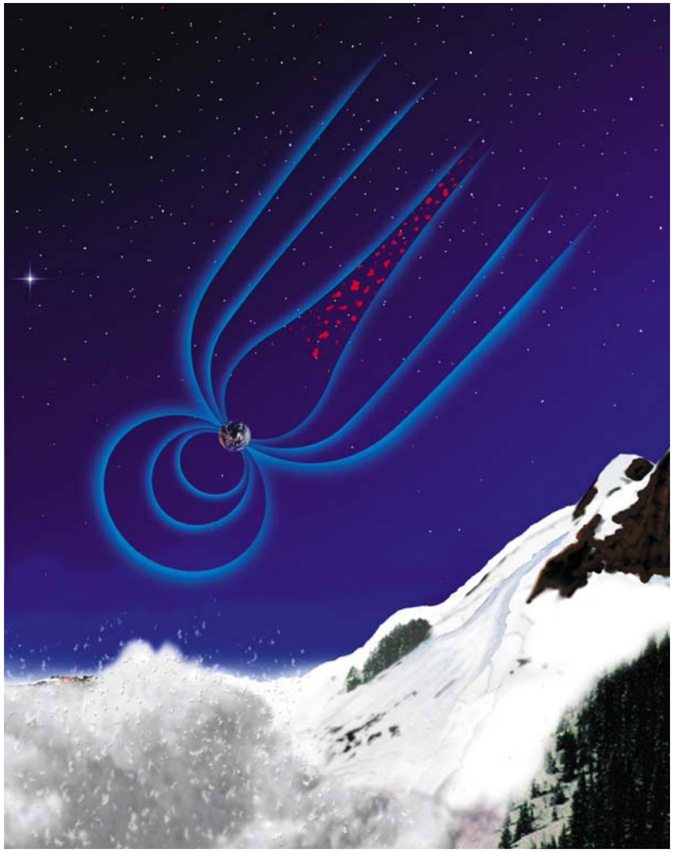
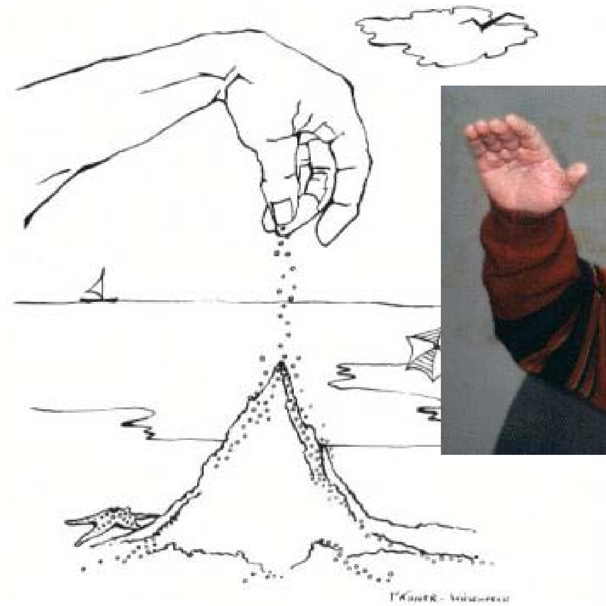
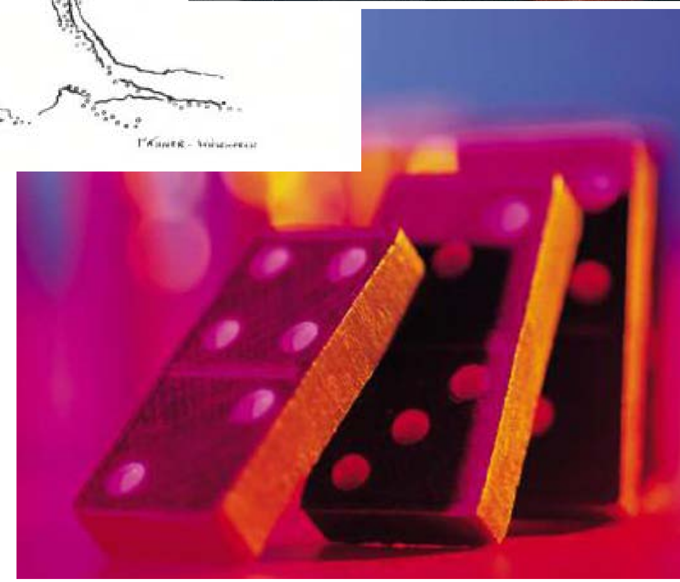


FIG. 2. Typical domain structures resulted from several local perturbations for a 100×100 array. Each cluster is triggered by a single perturbation.









We and some others in space physics were attracted by the SOC paradigm in mid 1990s, and worked both on forward problem of what sorts of plasma physics would map on to SOC, and also inverse problem of unambiguous identification of SOC in data. See **e.g Freeman and Watkins, Science 2002; Watkins, NPG, 2002; Chapman and Watkins, Space Sci Rev, 2001; Watkins et al, JASTP, 2001; Aschwanden, SOC In Astrophysics, Springer 2011 .** s

FROM FRACTALS TO PHYSICS ?

- Initial interest in extreme fluctuations in space physics and other environmental science problems, and need to compare paradigms like SOC to experimental data ...

...has now led to an interest in three related issues

AGGREGATION IS NOT NOISE

- Most familiar issue arises from the fact that a measured fluctuating quantity need not always be stationary and noise-like. Instead natural fluctuations may have been integrated or multiplied by the system's physics to create the observed variable(s).
- Aggregated fluctuations already have rather different properties to noise, some of which can be traps for the unwary.
- For example, the first passage time of even an "ordinary" Brownian random walk is <u>already</u> a heavy tailed random variable with infinite expectation value.
- Well studied, well known to **Mandelbrot, Bak et al,** and particularly familiar to today's audience ! Including it mainly because the Noah and Joseph effects are usually contrasted with white noise (as I did) and not random walks .... and to plug our **recent Warwick meeting** ...talks now online.

http://www2.warwick.ac.uk/fac/cross_fac/comcom/events/powerlawsandrareevents2012/programme/

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Warwick Mathematics Institute

Warwick Centre for Complexity Science

17-18 May 2012

Organisers: Colm Connaughton (Warwick Mathematics Institute and Warwick Centre for Complexity Science) and Nick Watkins (British Antarctic Survey)

Scientific Scope

The aggregation of random fluctuations in complex systems is a problem with aspects as abstract as the renormalisation group and as concrete as the risk industry. Classical statistics has given us the central limit theorem, describing the flow, under aggregation, of light-tailed fluctuations towards the Gaussian limit. In this context extreme events are rare, and are handled in the correspondingly mature framework of extreme value theory .

However, laboratory critical phenomena, fluid turbulence, and a wide range of socio-economic systems are increasingly recognised as giving rise to heavier-tailed distributions of fluctuations, in which "extreme" events are correspondingly much more common. Much progress has been made, notably through the use of additive models with alpha-stable ("Levy") distributions, or by multiplicative cascade processes, but many important open problems remain.

- How to get here

See also:

Mathematics Research Centre

Mathematical Interdisciplinary Research at Warwick (MIR@W)

Past Events

Past Symposia

Registration:
You can register for any of the symposia or workshops online. To see which registrations are currently open and to submit a registration.

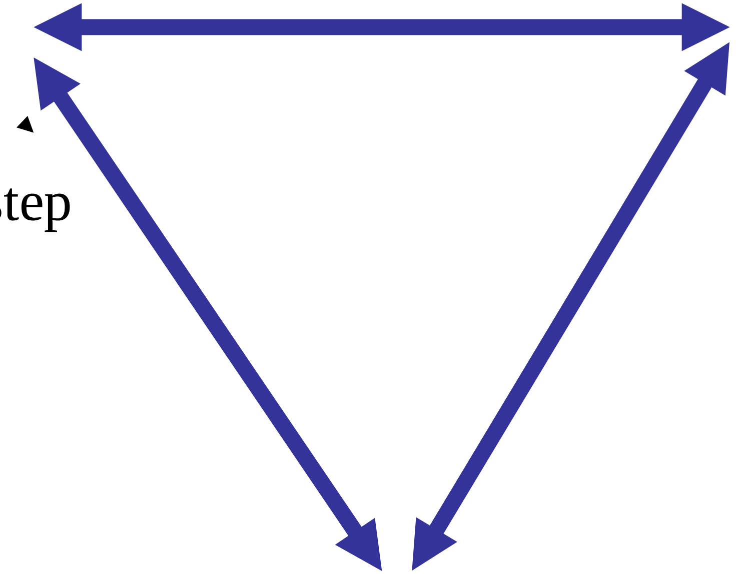
“Textbook” additive aggregation

Brownian random walk

$$\frac{dx}{dt} = \eta$$

η is stochastic iid step

Random walk picture



diffusion equation

$$\frac{\partial P(y,t)}{\partial t} = D\nabla^2 P(y,t)$$

$\Rightarrow P(y,t)$ is Gaussian

Kinetic description

Stability property: system looks the same under

$$t' = \frac{t}{\tau},\ y' = \frac{y}{\tau^{1/\alpha}} \text{ where } \alpha=2$$

$$P(y',t') = \tau^{1/\alpha} P(y,t)$$

$P(y,t)$ is Gaussian, a fixed point.

Stability property and central limit theorem

MEMORY IS NOT (ALWAYS) FRACTALITY

- A more subtle problem arises from the fact that a few of the very popular diagnostics are constructed to measure self-similarity , while most (e.g. R/S, DFA) in fact measure long-range dependence, so some confusion can arise when interpreting their outputs in systems where these two properties are not synonymous.
- Discussed in **Franzke et al, Phil Trans A, 2012; see also Mercik et al, "Enigma of self-similarity of fractional Levy stable motions", Acta Phys. Pol. B 34 3773 (2003).**

FRACTAL NOISE MODELS: HOW TO CHOOSE ?

- Several models modify the Brownian random walk, including those for "anomalous diffusion". **In Watkins et al, PRE, 2009** we attempted to present a classification, which needs some correction.
- Three particularly important classes of such models are:
  - additive and undamped, including fractional stable models, the fractional CTRW, and generalised shot noises
  - additive and mean reverting (damped) models like the Ornstein-Uhlenbeck process
  - multiplicative processes.

- Important to dispel the misconception that such models are ``just statistics'', as many embody a close correspondence with a physical scenario, which can be used as a guide when trying to choose the most suitable one to use, and which will bite back if you ignore it !

ADDITIVE UNDAMPED MODELS

- The variance of an additive stochastic process with no damping will tend to grow with time.
- It is thus a model of diffusion. Not so obvious that it will be a time series model.
- Recap links between Langevin and Fokker-Planck descriptions of diffusion, and the central limit theorem and scaling.

Brownian Case

Brownian random walk

$$\frac{dx}{dt} = \eta$$

η is stochastic iid step

Random walk

diffusion equation

$$\frac{\partial P(y,t)}{\partial t} = D\nabla^2 P(y,t)$$

$\Rightarrow P(y,t)$ is Gaussian

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Stability property and central limit theorem

- In Brownian case the three legs are (by now) very well studied and relate to each other.
- When we go beyond Brownian motion, it is not self-evident that can maintain all 3 properties ... and in fact when we look at the models that have been developed we will see that they don't.

Anomalous diffusion: CTRW

•If we choose to keep a (fractional) diffusion equation and lose the alpha-stability property we get Continuous Time Random Walk.

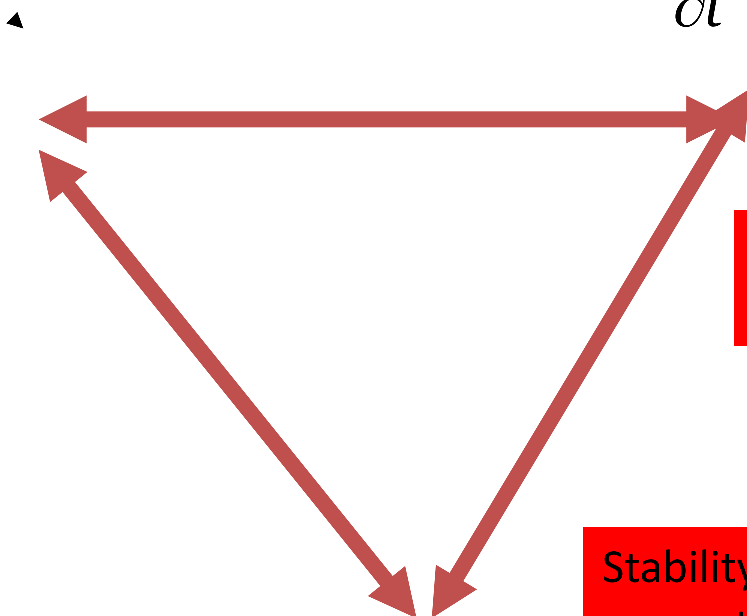
•Can model simplest, factorising, version of CTRW by specifying pdf of jump sizes and pdf of a random waiting time that elapses before next jump.
Has been used as a model of space physics data **[Zaslavsky et al, Physica A, 2007]**, but requires (non-uniform) waiting times to be defined on a uniformly sampled time series.

Anomalous diffusion & CTRW

fractional diffusion equation

Can simulate CTRW directly.

$$\frac{\partial^{\alpha H} P(y,t)}{\partial t^{\alpha H}} = D \nabla^{\alpha} P(y,t)$$



Random walk

Kinetic description

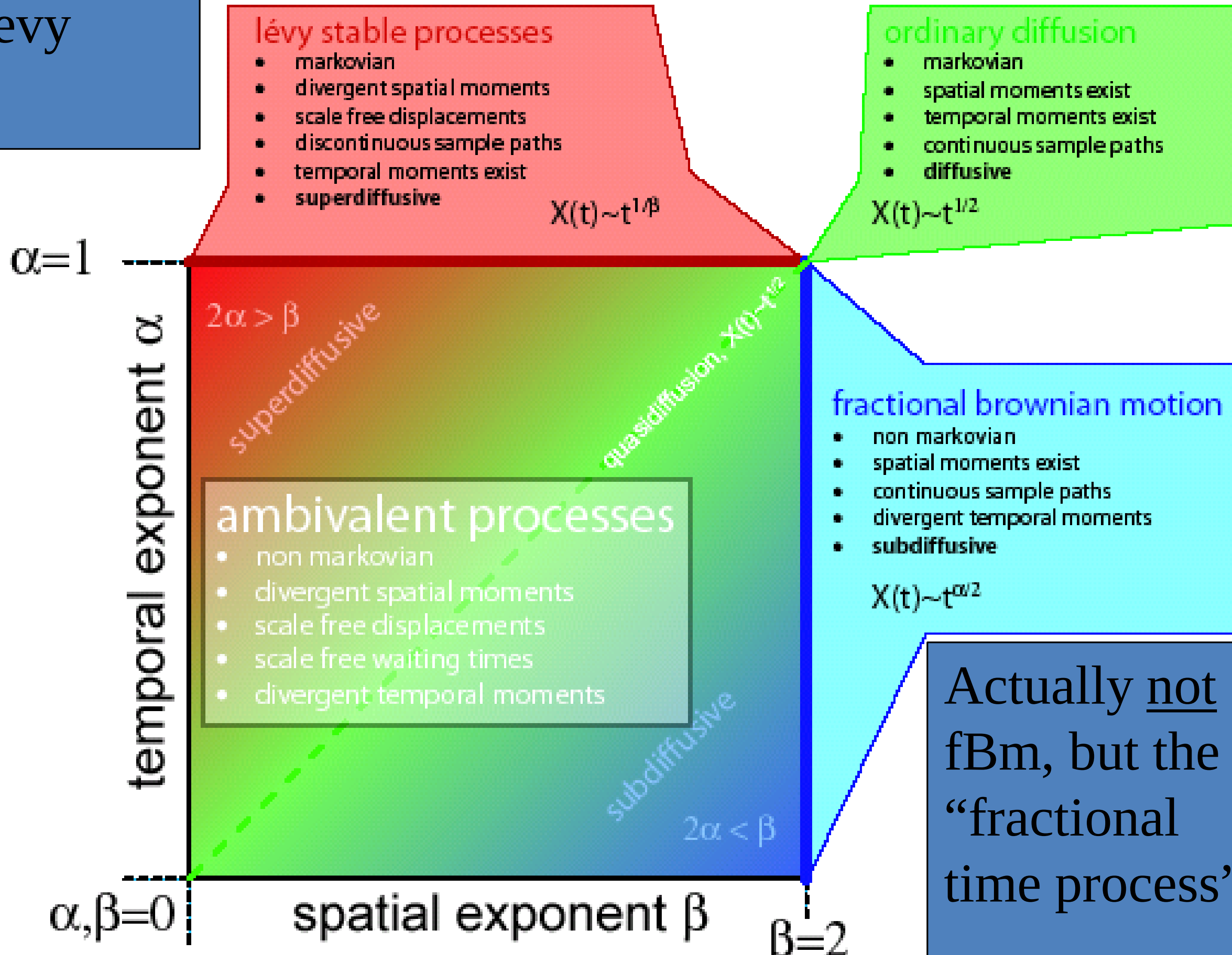
Keep

Stability property and central limit theorem

Lose

$P(y,t)$ is no longer $\alpha - stable$ but pdf has been obtained, see e.g. Kolokoltsev et al, J. Mathematical Sciences, 2001.

Careful ! α=time exponent, β=Levy exponent here



Actually not fBm, but the “fractional time process”

$$\nabla^2 p = \partial_t^{\alpha H} p$$

in our notation.

- From **Brockmann et al, Nature, 2006.**

Diffusion Equations

		Stable Process	CTRW
α	H	$H = 1/\alpha + d$	$H = d'/\alpha$
$\alpha = 2$	$H = 1/2$	Bachelier -Wiener Brownian Motion	
		$\nabla^2 p = \partial_t p$	
$0 < \alpha \leq 2$	$H = 1/\alpha$	Ordinary Levy Motion	
		$\nabla^\alpha p = \partial_t p$	
$\alpha = 2$	$0 \leq H \leq 1$	fBm	FTP
		?	$\nabla^2 p = \partial_t^{\alpha H} p$
$0 < \alpha \leq 2$	$0 \leq H \leq 1$	LFSM	FF CTRW
		?	$\nabla^\alpha p = \partial_t^{\alpha H} p$

- Revised from **Watkins et al, PRE, 2009**

Stable processes: linear fractional stable motion

•Conversely if we lose the diffusion equation but keep the alpha-stability property we get the family of fractional stable motions.

Has been proposed as a model of AE, **e.g. Watkins et al, Space Science Reviews, 2005.** Biggest deficiency is shape of the pdfs, which are heavier-tailed than reality (see **e.g. Rypdal & Rypdal, 2011**). Fix-up of truncated Levy approach less satisfactory than some others here.

Stable processes

Kinetic description

???

Lose

Can difference LFSM to give a fractional stable noise. Langevin-type descriptions with LFS noise driving do exist e.g. Magdziarz, Stochastic Models, 2007.

Random walks

Stability property and central limit theorem

Keep

$P(y,t)$ is still $\alpha - stable.$

Property preserved by use of a self-similar kernel.

$$Y_{H,\alpha}(t) = C_{H,\alpha}^{-1} \int_R \left((t-s)_+^{H-\frac{1}{\alpha}} - (-s)_+^{H-\frac{1}{\alpha}} \right) dL_\alpha(s)$$

Kinetic picture ?

Early work on fBm for example proposed that a diffusion equation with a diffusion coefficient with power law time dependence would describe it, see also **Watkins et al, 2009.**

$$\partial_t P = (\alpha H) t^{\alpha H - 1} D \, \partial^\alpha P$$

WRONG !, as this describes the pdf but not the correct temporal structure e.g. first passage time etc. See e.g. **Lim and Muniandy, PRE, 2002.**

Another process ?

The diffusion equation in the last slide, with nonlinear time dependence in its coefficient, seems to be a third process, neither Mandelbrot's fBm or the FPT ? Some light shed by **Lutz, PRE, 2001,** who derived it and a fractional Langevin equation from a concrete (system + heat bath) model. In his notation (where "α"= our "αH") FLE was:

$$M\ddot{x}(t) + M\gamma_\alpha \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\dot{x}(t) + U'(x) = \xi(t)$$

Diffusion Equations

		Stable Process	CTRW
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$\alpha = 2$	$H = 1/2$	Bachelier -Wiener Brownian Motion	
		$\nabla^2 p = \partial_t p$	
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		$\nabla^\alpha p = \partial_t p$	
$\alpha = 2$	$0 \leq H \leq 1$	fBm	FTP
		?	$\nabla^2 p = \partial_t^{\alpha H} p$
$0 < \alpha \leq 2$	$0 \leq H \leq 1$	LFSM	FF CTRW
		?	$\nabla^\alpha p = \partial_t^{\alpha H} p$

- After **Watkins et al, PRE, 2009**

Campbell's Theorem Approach

$$V_H(t) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{t} ds W(s) \left((t-s)^{H-\frac{1}{2}} \right)$$

Voss [in the Science of Fractal Images, 1988] called above "fBm", but interpreted as a sum of power law responses to shocks W(s). See also work by **Mandelbrot** on fractal sums of pulses and **Eliazar & Klafter** on generalised shot noises.

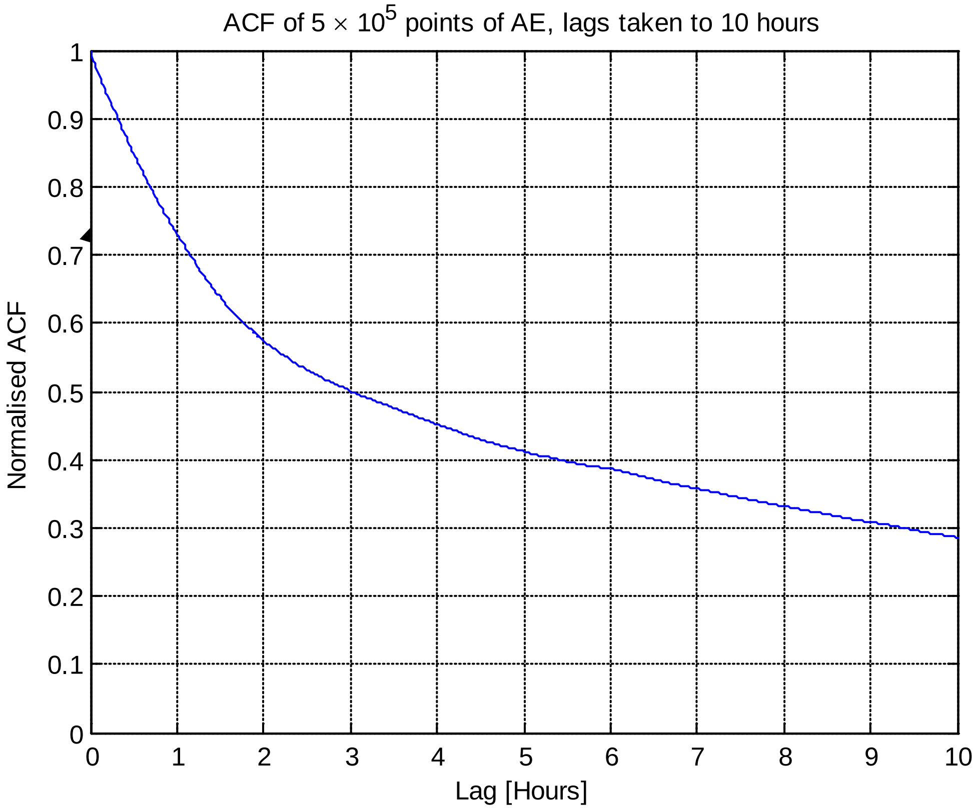
ADDITIVE DAMPED MODELS

Mean reversion

- A key step in modifying additive random walk for physical applications was introduction of a damping time scale.
- In physics we have Ornstein-Uhlenbeck model of damped Brownian motion
- In stochastics we have first order autoregressive process AR(1):

$$X_n = \lambda X_{n-1} + \epsilon$$

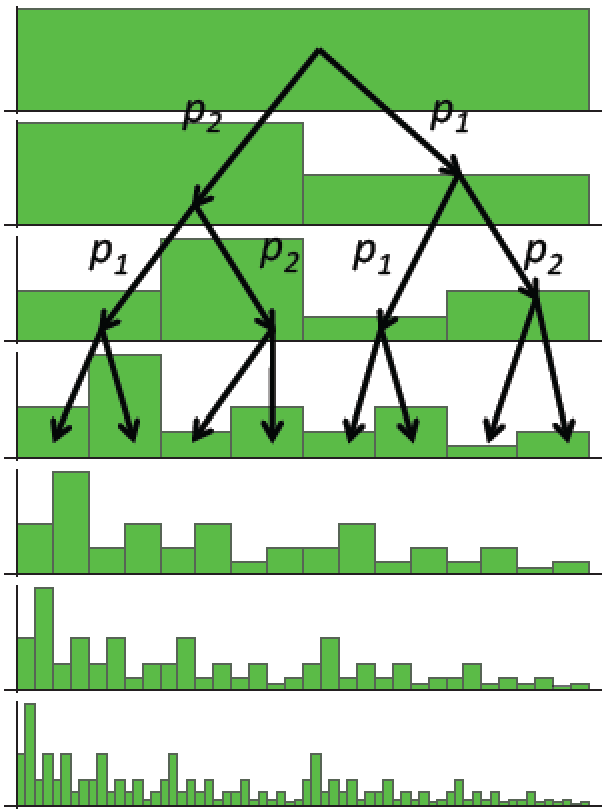
AE



•Strongly indicated for AE which has ~ exponential decay in ACF on short time scales less than a few hours.

MULTIPLICATIVE MODELS

Cascade processes



Many systems have aggregation, but not by an additive route.

Classic example is turbulence.

One indicator can be the presence of ~ lognormality in pdf.

Modelling AE

An interesting recent synthesis of these approaches has been combination of mean reversion and a multifractal noise by **Rypdal and Rypdal, 2011** to model AE.

As in finance a log transformation was first employed to give a series with near-stationary increments.

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- Pitfalls: 1. Walks are not noises
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Organisers: Colm Connaughton (Warwick Mathematics Institute and Warwick Centre for Complexity Science) and Nick Watkins (British Antarctic Survey)

Scientific Scope

The aggregation of random fluctuations in complex systems is a problem with aspects as abstract as the renormalisation group and as concrete as the risk industry. Classical statistics has given us the central limit theorem, describing the flow, under aggregation, of light-tailed fluctuations towards the Gaussian limit. In this context extreme events are rare, and are handled in the correspondingly mature framework of extreme value theory .

However, laboratory critical phenomena, fluid turbulence, and a wide range of socio-economic systems are increasingly recognised as giving rise to heavier-tailed distributions of fluctuations, in which "extreme" events are correspondingly much more common. Much progress has been made, notably through the use of additive models with alpha-stable ("Levy") distributions, or by multiplicative cascade processes, but many important open problems remain.

- How to get here

See also:

Mathematics Research Centre

Mathematical Interdisciplinary Research at Warwick (MIR@W)

Past Events

Past Symposia

Registration:
You can register for any of the symposia or workshops online. To see which registrations are currently open and to submit a registration.