MTH4100 Calculus I

Week 1 (mainly Thomas' Calculus Section 1.1)

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What is Calculus?

"advanced algebra and geometry":
setting up Mathematics as a formal language

- fundamental: real numbers
- study of functions of real variables
  - one real variable (Calculus I)
  - many variables (Calculus II)
- geometric view: graph of a function
  - continuity properties
  - slope $\leftrightarrow$ derivative
  - area $\leftrightarrow$ integral
- many techniques, based on algebraic manipulations
- many applications in all branches of modern society

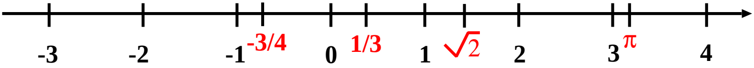
(next level of mathematical abstraction is called analysis)

Real numbers and the real line

think of the real numbers, e.g., as all decimals

examples: $-\frac{3}{4} = -0.7500\ldots \quad ; \quad \frac{1}{3} = 0.333\ldots \quad ; \quad \sqrt{2} = 1.4142\ldots$

The real numbers $\mathbb{R}$ can be represented as points on the real line:



- three fundamental properties of real numbers
  - algebraic: formalisation of rules of calculation (addition, subtraction, multiplication, division)
    example: $2(3+5) = 2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16$
  - order: inequalities (geometric picture: see the real line!)
    example: $-\frac{3}{4} < \frac{1}{3} \quad \Rightarrow -\frac{1}{3} < \frac{3}{4}$
  - completeness: there are "no gaps" on the real line

Algebraic properties

1. algebraic properties of the reals for addition ($a, b, c \in \mathbb{R}$):

(A1) $a + (b + c) = (a + b) + c$ *associativity*

(A2) $a + b = b + a$ *commutativity*

(A3) there is a 0 such that $a + 0 = a$ *identity*

(A4) there is an x such that $a + x = 0$ *inverse*

why these rules? They define an algebraic structure (commutative group).
define analogous algebraic properties for multiplication:

(M1) $a(bc) = (ab)c$

(M2) $ab = ba$

(M3) there is a 1 such that $a\,1 = a$

(M4) there is an x such that $a\,x = 1$ (for $a \neq 0$)

connect multiplication with addition: (D) $a(b + c) = ab + ac$ *distributivity*
(These 9 rules define an algebraic structure called a *field*.)

Order: the real number line

2. Order properties of the reals:

(O1) for any $a, b \in \mathbb{R}$, $a \leq b$ or $b \leq a$ *totality of ordering I*

(O2) if $a \leq b$ and $b \leq a$ then $a = b$ *totality of ordering II*

(O3) if $a \leq b$ and $b \leq c$ then $a \leq c$ *transitivity*

(O4) if $a \leq b$ then $a + c \leq b + c$ *order under addition*

(O5) if $a \leq b$ and $0 \leq c$ then $a\,c \leq b\,c$ *order under multiplication*

Rules for inequalities

some useful rules for calculations with inequalities (see exercises!):

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

these rules can all be "proven" by using (O1) to (O5): 1. to 3. follow straightforwardly, 4. to 6. require more work

Subsets of the real numbers $\mathbb{R}$

3. Completeness property can be understood by the following construction of the real numbers: (! using set notation !)

Start with "counting numbers" $1, 2, 3, \ldots$

- $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ natural numbers

  $\rightarrow$ can we solve $a + x = b$ for x?
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ integers

  $\rightarrow$ can we solve $ax = b$ for x?
- $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z},\ q \neq 0\}$ rational numbers

  $\rightarrow$ can we solve $x^2 = 2$ for x?
- $\mathbb{R}$ real numbers

  example: positive solution to the equation $x^2 = 2$ is $\sqrt{2}$

  This is an irrational number whose decimal representation is not eventually repeating: $\sqrt{2} = 1.414\ldots$ Another example is $\pi = 3.141\ldots$

$$\Rightarrow \boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}}$$

$\mathbb{Q}$ has "holes"

In fact, one has to "prove" this:

Theorem

$x^2 = 2$ *has no solution* $x \in \mathbb{Q}$

The real numbers $\mathbb{R}$ are complete in the sense that they correspond to all points on the real line, i.e., there are no "holes" or "gaps", whereas the rationals have "holes" (namely the irrationals)

(see your textbook Appendix 4 for details; "proof" of completeness of $\mathbb{R}$ covered in MTH5104 Convergence and Continuity, 2nd year "analysis" module)

Revision of Lecture 1

Properties of real numbers $\mathbb{R}$:

- algebraic: rules of calculation
- order: inequalities
- completeness: "no gaps"

$\sqrt{2}$ is irrational

Theorem

$x^2 = 2$ *has no solution* $x \in \mathbb{Q}$

Proof.

Assume there is an $x \in \mathbb{Q}$ with $x^2 = 2$. This must be of the form $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$, and we can assume that **p and q have no common factors** (otherwise cancel them).

$x^2 = 2$ then implies that $(\frac{p}{q})^2 = 2$, or $\boxed{p^2 = 2q^2}$, so p^2 is even.

However, p^2 even implies that **p is even** (...requires proof...).

Write $p = 2p_1$, so that $p^2 = (2p_1)^2$, or $4p_1^2 = 2q^2$, or $\boxed{2p_1^2 = q^2}$.

This implies that q^2 is even, so **q is even** as well.

We have now shown that both p and q must be even, so they **share a common factor 2**.

This is a contradiction! Therefore the assumption must be wrong. □

Definitions, Theorems, Proofs, ...

You have just seen a Theorem with Proof.

University mathematics is built upon

- basic properties (Definitions, Axioms)
- statements deduced from these
  (Lemma, Proposition, Theorem, Corollary, ...)
  in form of *proofs*!

example: The technique in the previous proof is called Proof by Contradiction.

Many different ones to come! Details about the logic behind proofs, e.g., in MTH5117 Mathematical Writing.

This formal framework is illustrated in Calculus 1 by many examples, exercises, applications, ...

Intervals

Definition

A subset of the real line is called an interval if it contains at least two numbers and all the real numbers between any two of its elements.

examples:

- $x > -2$ defines an *infinite interval*; geometrically, it corresponds to a *ray* on the real line
- $3 \leq x \leq 6$ defines a *finite interval*; geometrically, it corresponds to a *line segment* on the real line

So we can distinguish between two basic types of intervals – let's further classify:

Types of Intervals

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Finding intervals of numbers

Solve inequalities to find intervals of $x \in \mathbb{R}$:

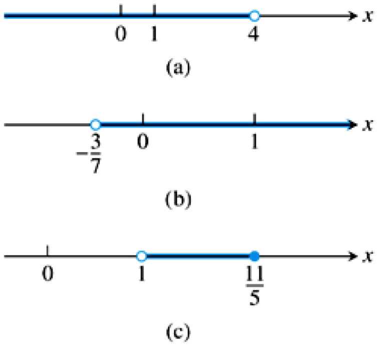
$$(a) \quad \begin{aligned} 2x - 1 &< x + 3 \\ 2x &< x + 4 \\ x &< 4 \end{aligned}$$

$$(b) \quad \begin{aligned} -\frac{x}{3} &< 2x + 1 \\ -x &< 6x + 3 \\ -\frac{3}{7} &< x \end{aligned}$$

(c) $\frac{6}{x-1} \geq 5$: must hold $x > 1$!

$$\begin{aligned} 6 &\geq 5x - 5 \\ \frac{11}{5} &\geq x \end{aligned}$$

solution sets on the real line:



Absolute Value

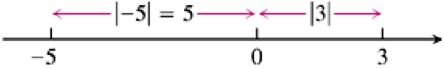
Definition

The absolute value (or *modulus*) of a real number x is

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} .$$

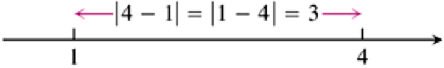
geometrically, $|x|$ is the *distance* between x and 0

example:



$|x - y|$ is the distance between x and y

example:



an alternative definition of $|x|$ is

$$|x| = \sqrt{x^2} \quad ,$$

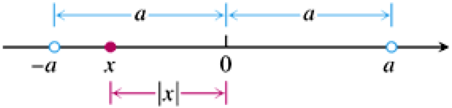
since taking the square root always gives a *non-negative* result!

Inequalities with $|x|$

$|x|$ in an inequality:

$$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

distance from x to 0 is less than $a > 0 \Leftrightarrow x$ must lie between a and $-a$



absolute value properties:

1. $|-a| = |a|$
2. $|ab| = |a|\,|b|$
3. $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$
4. $|a+b| \leq |a| + |b|$, the *triangle inequality*

prove these statements!

Revision of Lecture 2

Definitions, Theorems, Proofs:

- theorem and proof - example: irrationality of $\sqrt{2}$
- definition: interval; and examples: (a, b), $[a, b]$, $[a, \infty)$, etc.
- another definition: absolute value $|x|$ and some properties

Some simple proofs

key idea: use $|x| = \sqrt{x^2}$

1. Proof of $|-a| = |a|$:

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$$

Note: we have used a direct proof: we started on the left hand side (LHS) of the equation and transformed it step by step until we have arrived at the right hand side (RHS)

2. Proof of $|ab| = |a|\,|b|$:

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2 b^2} = \sqrt{a^2}\sqrt{b^2} = |a|\,|b|$$

3. Proof of $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$:

$$\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$$

Proof of the triangle inequality

4. Proof of $|a+b| \leq |a| + |b|$: use a little trick and prove instead:

$$|a+b|^2 \leq (|a| + |b|)^2$$

$$\begin{aligned} |a+b|^2 &= (a+b)^2 \qquad (|x| = \pm x \text{ yields } |x|^2 = x^2) \\ &= a^2 + 2ab + b^2 \\ &\leq a^2 + 2|a|\,|b| + b^2 \qquad (\text{because } ab \leq |ab| = |a||b|) \\ &= |a|^2 + 2|a|\,|b| + |b|^2 \qquad (\text{see above}) \\ &= (|a| + |b|)^2 \end{aligned}$$

now take the square root and observe that the arguments of both roots are positive – we are done.

Further properties

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$
6. $|x| < a$ if and only if $-a < x < a$
7. $|x| > a$ if and only if $x > a$ or $x < -a$
8. $|x| \leq a$ if and only if $-a \leq x \leq a$
9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

note: "if and only if" is often abbreviated by the sign "$\Leftrightarrow$"

examples

(a) $|2x - 3| \leq 1$

(a)

(b) $|2x - 3| \geq 1$

(b)

Three important inequalities

Triangle inequality

$$|a+b| \leq |a| + |b|$$

arithmetic mean: $\frac{1}{2}(a+b)$; geometric mean $\sqrt{ab}$

Arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \qquad \text{for } a, b \geq 0$$

Cauchy-Schwarz inequality

$$(ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$$

Proof of the arithmetic-geometric mean inequality

- multiply inequality by 2 and square:

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \quad \Leftrightarrow 4ab \leq (a+b)^2$$

- use direct proof: start on RHS until the LHS is obtained

$$\begin{aligned} (a+b)^2 &= a^2 + 2ab + b^2 \\ &= a^2 + 2ab + 2ab - 2ab + b^2 \\ &= 4ab + (a-b)^2 \quad , \quad (a-b)^2 \geq 0 \text{ and therefore} \\ &\geq 4ab \end{aligned}$$

Proof of the Cauchy-Schwarz inequality

- Use direct proof: start on one side until the other side is obtained

$$(ac+bd)^2 \le (a^2+b^2)(c^2+d^2)$$

decide which side:

$$\begin{aligned} \text{LHS:} \quad (ac+bd)^2 &= a^2c^2+2abcd+b^2d^2 \\ \text{RHS: } (a^2+b^2)(c^2+d^2) &= a^2c^2+b^2c^2+a^2d^2+b^2d^2 \end{aligned}$$

- Start on RHS and work it out:

$$\begin{aligned} (a^2+b^2)(c^2+d^2) &= a^2c^2+2abcd+b^2d^2 \\ &\quad +b^2c^2-2abcd+a^2d^2 \\ &= (ac+bd)^2+(bc-ad)^2 \\ &\ge (ac+bd)^2 \end{aligned}$$

Q.E.D. (quod erat demonstrandum)

Reading Assignment

Read

Thomas' Calculus, Chapter 1.2:

Lines, Circles, and Parabolas