Queen Mary
University of London

MTH4100 Calculus I

Lecture notes for Week 10

Thomas' Calculus, Sections 5.2 to 5.6

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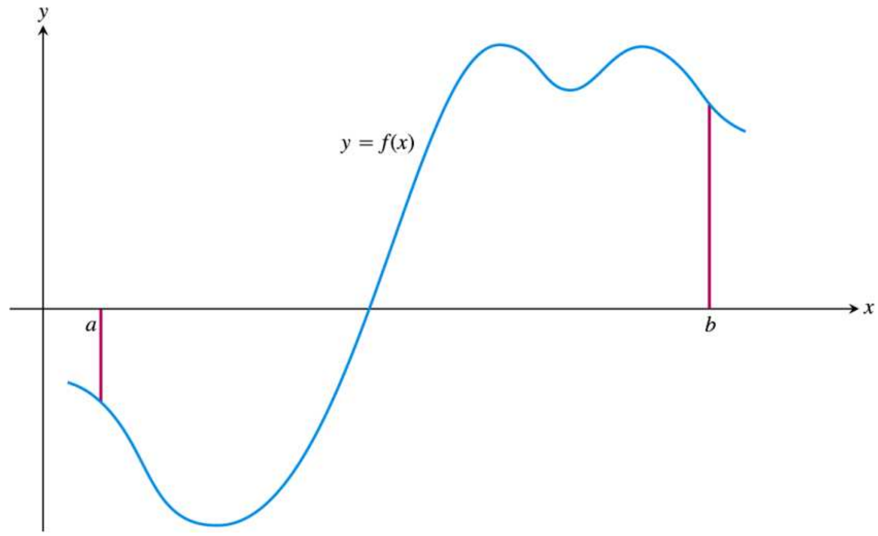
Lecture 25

Riemann sums and definite integral

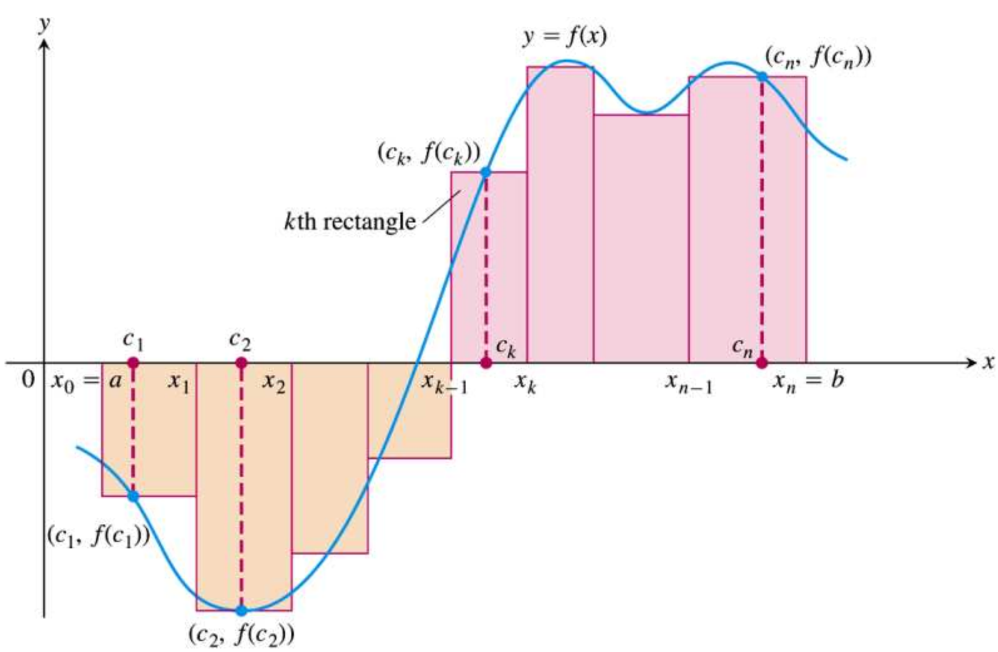
Consider a typical continuous function over $[a, b]$:



Partition $[a, b]$ by choosing $n - 1$ points between a and b:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\,,$$

i.e., $\Delta x_k = x_k - x_{k-1}$, the width of the subinterval $[x_{k-1}, x_k]$, may vary.
Choose $c_k \in [x_{k-1}, x_k]$ and construct rectangles:



The resulting sums

$$S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

are called *Riemann sums for* f *on* $[a, b]$.
Then choose finer and finer partitions by taking the limit such that the width of the *largest subinterval* goes to zero.
For a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ we write $||P||$ (called "norm") for the width of the largest subinterval.

DEFINITION The Definite Integral as a Limit of Riemann Sums

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number I is the **definite integral of f over $[a, b]$** and that I is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

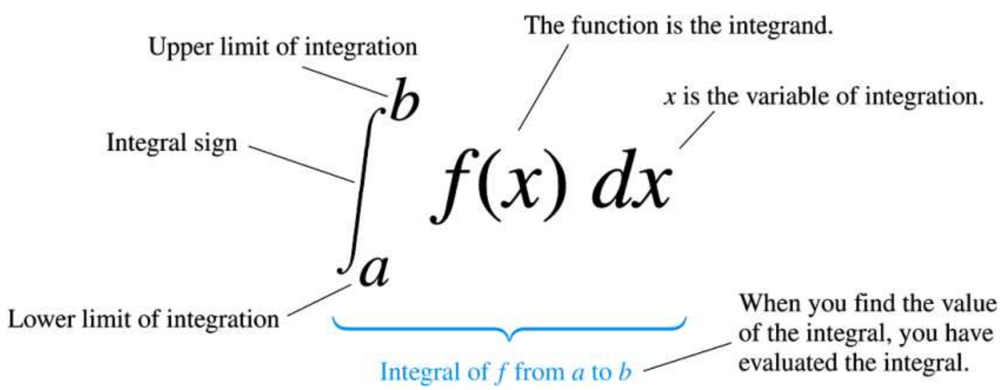
Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - I \right| < \epsilon.$$

shorthand notation:

$$I = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)dx$$

with



note:

$$\int_a^b f(t)dt = \int_a^b f(x)dx \;, \text{ etc.}$$

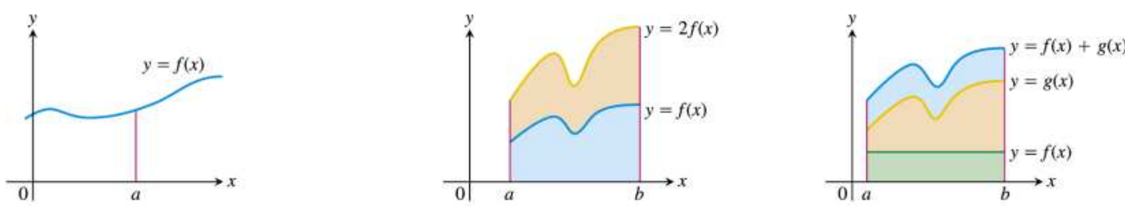
THEOREM 1 The Existence of Definite Integrals
A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

(idea of proof: check convergence of upper/lower sums; see p.345 of book for further details)
example of a nonintegrable function on [0.1]:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

upper sum is always 1; lower sum is always 0 $\Rightarrow \int_0^1 f(x)dx$ does not exist!

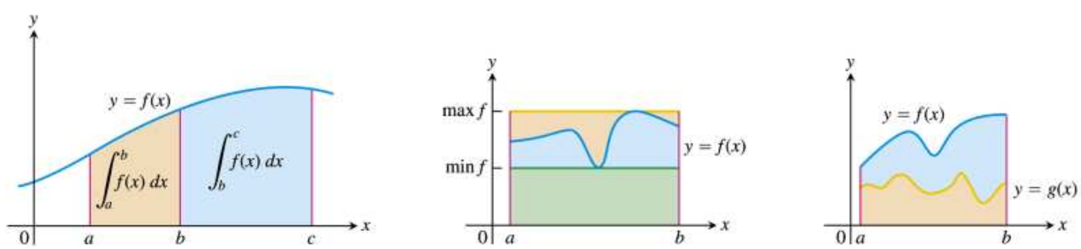
Theorem 2 *For integrable functions f, g on $[a, b]$ the definite integral satisfies the following rules:*



(a) *Zero Width Interval:* $\int_a^a f(x)\,dx = 0.$ (The area over a point is 0.)

(b) *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$ (Shown for $k = 2$.)

(c) *Sum:* $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ (Areas add)



(d) *Additivity for definite integrals:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) *Max-Min Inequality:* $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$

(f) *Domination:* $f(x) \ge g(x)$ on $[a, b]$ $\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

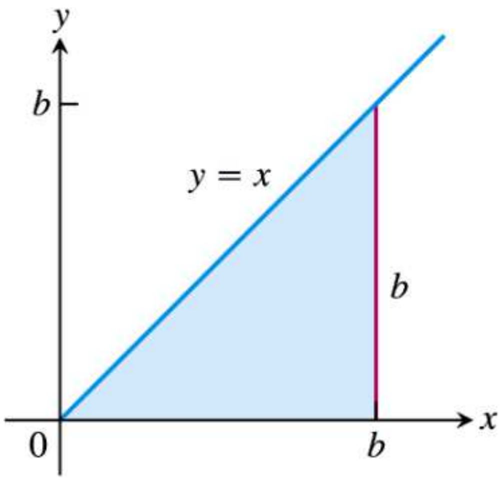
and *(g) order of integration:*

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

(for idea of proof of (b) to (f) see book p.348; (a), (g) are definitions!)

Area under the graph and mean value theorem

example: $f(x) = x$, $a = 0$, $b > 0$



- *area* $A = \frac{1}{2}b^2$
- definition of integral:
  choose $x_k = kb/n$ with right endpoints c_k

$$
\begin{aligned}
I &= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{kb}{n} \cdot \frac{b}{n} \\
&= \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{k=1}^{n} k \\
&= \lim_{n\to\infty} \frac{b^2}{n^2} \frac{n(n+1)}{2} = \frac{b^2}{2}
\end{aligned}
$$

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DEFINITION Area Under a Curve as a Definite Integral
If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\, dx.$$

Consider the (arithmetic) *average* of n function values on $[a, b]$:

$$\frac{1}{n}\sum_{k=1}^{n} f(c_k) = \frac{1}{n\Delta x}\sum_{k=1}^{n} f(c_k)\Delta x \rightarrow \frac{1}{b-a}\int_a^b f(x)dx \ (n \rightarrow \infty)$$

DEFINITION The Average or Mean Value of a Function
If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

example: $f(x) = x$, $x \in [0, b]$ (see above)

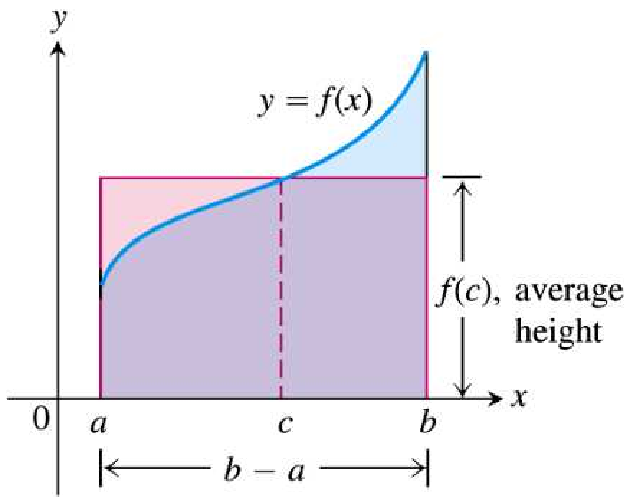
$$\text{av}(f) = \frac{1}{b-0}\int_0^b x dx = \frac{1}{b}\left.\frac{x^2}{2}\right|_0^b = \frac{b^2}{2b} = \frac{b}{2}$$

Theorem 3 (The mean value theorem for definite integrals) *If f is continuous on $[a, b]$, then there is a $c \in [a, b]$ with*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx\ .$$

Interpretation, loosely speaking: "f assumes its average value somewhere on $[a, b]$."

geometrical meaning:



(proof: see book p.357; not hard; based on max-min-inequality for integrals and intermediate value theorem for continuous functions)
example for applying the mean value theorem for integrals:
Let f be continuous on $[a, b]$ with $a \neq b$ and

$$\int_a^b f(x)dx = 0 \ .$$

Show that $f(x) = 0$ at least once in $[a, b]$.
Solution: According to the last theorem, there is a $c \in [a, b]$ with

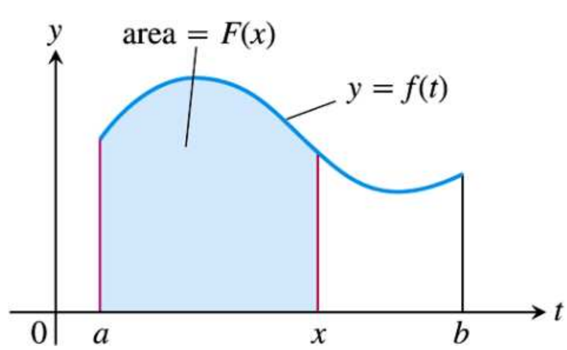
$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx = 0 \ .$$

The fundamental theorem of calculus

For a continuous function f, define

$$F(x) = \int_a^x f(t)dt \ .$$

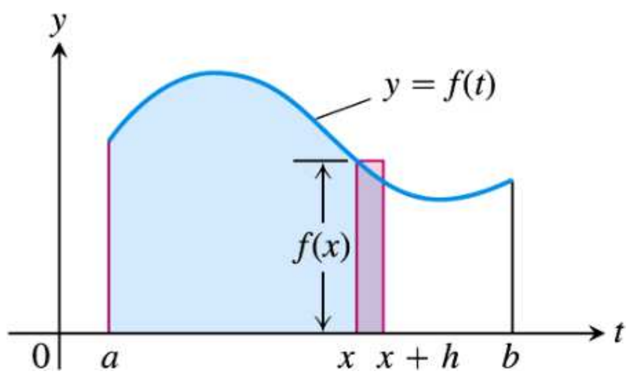
Geometric interpretation:



Compute the difference quotient:

$$
\begin{aligned}
\frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) \\
\text{(additivity rule and see figure below)} &= \frac{1}{h}\int_x^{x+h} f(t)dt \\
\text{(mean value theorem for definite integrals)} &= f(c)
\end{aligned}
$$

for some c with $x \le c \le x+h$.



Since f is continuous,

$$\lim_{h\to 0} f(c) = f(x)$$

and therefore

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)\ .$$

We have just proven (except a little detail - which one?)

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

examples:

1.

$$\frac{d}{dx}\int_a^x \frac{1}{1+4t^3}dt = \frac{1}{1+4x^3}$$

2. Find

$$\frac{d}{dx}\int_2^{x^2} \cos t\, dt\ :$$

Define

$$y = \int_2^u \cos t\, dt \text{ with } u = x^2$$

Apply the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= \left(\frac{d}{du}\int_2^u \cos t\, dt\right)\cdot\frac{du}{dx} \\ &= \cos u \cdot 2x \\ &= 2x\cos x^2 \end{aligned}$$

We know that

$$\int_a^x f(t)dt = G(x)$$

is an antiderivative of f, as $G'(x) = f(x)$, see theorem above.
The most general antiderivative is $F(x) = G(x) + C$ (why?). We thus have

$$\begin{aligned} F(b) - F(a) &= (G(b) + C) - (G(a) + C) \\ &= G(b) - G(a) \\ &= \int_a^b f(t)dt - \int_a^a f(t)dt \\ \text{(zero width interval rule)} &= \int_a^b f(t)dt\ . \end{aligned}$$

We have just proven (supplemented by considering F, G at the boundary points a, b)

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2
If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

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Recipe to calculate $\int_a^b f(x)dx$:

1. Find an antiderivative F of f
2. Calculate $F(b) - F(a)$

Notation:

$$F(b) - F(a) = F(x)|_a^b$$

example:

$$\begin{aligned}\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left(x^{3/2} + \frac{4}{x}\right)\Bigg|_1^4 \\ &= \left(4^{3/2} + \frac{4}{4}\right) - \left(1^{3/2} + \frac{4}{1}\right) \\ &= 4\end{aligned}$$

Fundamental theorem of calculus: summary

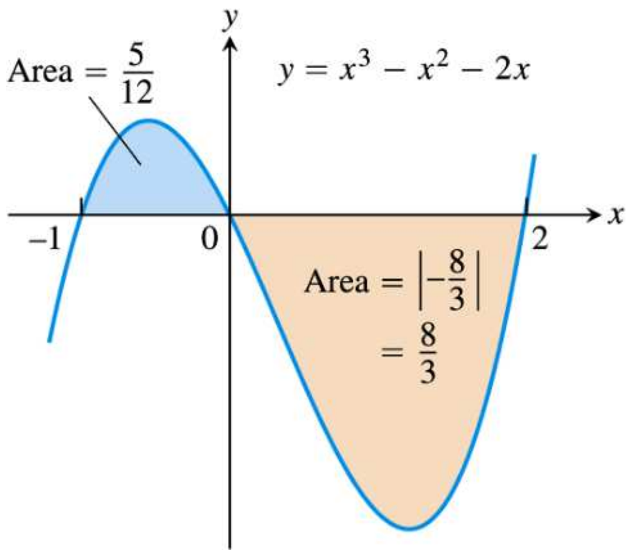
$$\frac{d}{dx}\int_a^x f(t)dt = \frac{dF}{dx} = f(x)$$

$$\int_a^x f(t)dt = \int_a^x \frac{dF}{dt}dt = F(x) - F(a)$$

Processes of integration and differentiation are "inverses" of each other!

Finding total areas

example:



To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate over each subinterval.
3. Add the *absolute* values of the integrals.

example continued:

$$f(x) = x^3 - x^2 - 2x\,,\ -1 \le x \le 2$$

1. $f(x) = x(x^2 - x - 2) = x(x+1)(x-2)$: zeros are -1, 0, 2
2. $$\int_{-1}^{0} (x^3 - x^2 - 2x)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} = \frac{5}{12}$$
   $$\int_{0}^{2} (x^3 - x^2 - 2x)dx = \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{0}^{2} = -\frac{8}{3}$$
3. $A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}$

The substitution rule

motivation: develop more general techniques for calculating antiderivatives
Recall the chain rule for $F(g(x))$:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

If F is an antiderivative of f, then

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x)$$

Now compute

$$\begin{aligned}
\int f(g(x))g'(x)dx &= \int \left(\frac{d}{dx}F(g(x))\right) dx \\
\text{(fundamental theorem)} &= F(g(x)) + C \\
(u = g(x)) &= F(u) + C \\
\text{(fundamental theorem)} &= \int F'(u)du \\
&= \int f(u)du
\end{aligned}$$

We have just proven

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

method for evaluating

$$\int f(g(x))g'(x)dx \ :$$

1. Substitute $u = g(x)$, $du = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace $u = g(x)$.

example: Evaluate

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz \ :$$

1. Substitute $u = z^2 + 5$, $du = 2z\,dz$:

$$\int \frac{2z}{\sqrt[3]{z^2+5}}dz = \int u^{-1/3}du$$

2. Integrate:

$$\int u^{-1/3}du = \frac{3}{2}u^{2/3} + C$$

3. Replace $u = z^2 + 5$:

$$\int \frac{2z}{\sqrt[3]{z^2+5}} dz = \frac{3}{2}(z^2+5)^{2/3} + C$$

Transform integrals by using trigonometric identities.

example: Evaluate $\int \sin^2 x \, dx$:

Use half-angle formula $\sin^2 x = (1 - \cos 2x)/2$ to write

$$\begin{aligned} \int \sin^2 x \, dx &= \int \frac{1}{2}(1 - \cos 2x) dx \\ &= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 2x \, dx \\ &= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \end{aligned}$$

Move on to substitution in definite integrals:

Theorem 6 *If g is continuous on $[a, b]$ and f is continuous on the range of g, then*

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du \, .$$

(note that $u = g(x)$! proof straightforward, see book p.377)

example: Evaluate $\int_{-1}^{1} 3x^2 \sqrt{x^3+1} dx$.

Substitute $u = x^3 + 1$, $du = 3x^2 dx$.

$x = -1$ gives $u = (-1)^3 + 1 = 0$; $x = 1$ gives $u = 1^3 + 1 = 2$, and we obtain

$$\begin{aligned} \int_{-1}^{1} 3x^2 \sqrt{x^3+1} dx &= \int_0^2 \sqrt{u} du \\ &= \frac{2}{3} u^{3/2} \Big|_0^2 \\ &= \frac{2}{3} 2^{3/2} - 0 \\ &= \frac{4\sqrt{2}}{3} \end{aligned}$$