E_8

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1 Introduction

This is the first talk in a projected series of five, which has two main aims. First, to describe some research I did over the summer. And second, to introduce some important pieces of mathematics to (especially) our research students. The research topic is in effect a marriage between the octonions and the Leech lattice: they have been flirting for years, but I can now report that the marriage has been consummated. But our main purpose is to meet the wedding guests: the friends and relatives of the bride and groom. As you would expect, they are a motley crew, and include a number of colourful characters, particularly eccentric uncles and the like.

Although the general title of my talks is ‘Construction of simple groups’, in fact I want to de-emphasise the group theory, and talk mainly about lattices, algebras and codes. Much of what I shall say is not new, but in many cases the presentation may be non-standard. In other words, there will be something old, and something new, something borrowed—and maybe even something blue.

2 2-dimensional reflection groups

Let us begin by meeting some of the bride’s older relatives, the *dihedral groups*.

Reflection in a vector r is the map which negates r and fixes everything perpendicular to r. Thus

$$x \mapsto x - 2\frac{(x.r)}{(r.r)}r$$

In terms of complex numbers, reflection in 1 is the map

$$x \mapsto -\overline{x}$$

as you can easily check: $1 \mapsto -1$ and $i \mapsto i$.

To get the reflection in r (without loss of generality, $r\overline{r} = 1$), first multiply by $\overline{r}$ to map r to 1, then reflect in 1, and then multiply by r to map 1 back to r. Thus

$$z \mapsto -r\overline{\overline{r}z} = -r\overline{z}r$$

The product of these two reflections is the rotation

$$z \mapsto rzr$$

Examples

	triangle	square	pentagon	hexagon
called:	A_2	B_2	H_2	G_2
the dihedral groups:	D_6	D_8	D_{10}	D_{12}
generating reflections:	$1, e^{\pi i/3}$	$1, e^{\pi i/4}$	$1, e^{\pi i/5}$	$1, e^{\pi i/6}$
in real numbers:	$(2, 0)$	$(2, 0)$	$(2, 0)$	$(2, 0)$
	$(-1, \sqrt{3})$	$(\sqrt{2}, \sqrt{2})$	$((1+\sqrt{5})/2, \sqrt{(5-\sqrt{5})/4})$	$(\sqrt{3}, 1)$

The crystallographic restriction is that we only consider shapes which tessellate the plane, i.e. the cases A_2, B_2 and G_2. In these cases we can choose the lengths of the roots (i.e. reflecting vectors) in such a way that they span a lattice $\mathbb{Z}$[roots].

A_2 and G_2 both give the lattice $\mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3}$. The roots of A_2 are $\pm 1, \pm\omega, \pm\overline{\omega}$. These are also the short roots of G_2. The long roots of G_2 are $\pm(1-\omega), \pm(\omega - \overline{\omega}), \pm(\overline{\omega} - 1)$. Or take these as the short roots, and multiply the others by 3 to make them the long roots!

B_2 gives $\mathbb{Z}[i]$ with short roots $\pm 1, \pm i$ and long roots $\pm 1 \pm i$ (or short roots $\pm 1 \pm i$ and long roots $\pm 2, \pm 2i$).

3 3-dimensional reflection groups

Classification of reflection groups can be done by induction on the dimension. There is no extension of G_2 to three dimensions, and an essentially unique extension of each of A_2, B_2 and H_2.

	tetrahedron	cube or octahedron	dodecahedron or icosahedron
	A_3	B_3	H_3
group:	S_4	$2 \times S_4$	$2 \times A_5$
coordinates:		$8V : (\pm1, \pm1, \pm1)$	$20V : (\pm1, \pm1, \pm1), (0, \pm\tau, \pm\sigma)$
		$12E : (\pm1, \pm1, 0)$	$30E : (\pm2, 0, 0), (\pm1, \pm\sigma, \pm\tau)$
		$6F : (\pm1, 0, 0)$	$12F : (0, \pm1, \pm\tau)$
			where $\sigma = (\sqrt{5} - 1)/2, \tau = (\sqrt{5} + 1)/2$

In the case of B_3 there are 6 short roots $(\pm1, 0, 0)$ and 12 long roots $(\pm1, \pm1, 0)$. If we double the length of the short roots this time we get a different shaped configuration, called C_3.

Quaternions $\mathbb{H} = \mathbb{R}[i, j, k]$ where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ can be used to describe these groups and the corresponding lattices. Which means we might as well go to four dimensions while we're about it.

4 4-dimensional reflection groups

H_3 extends uniquely, to H_4, but A_3 extends in two ways, to A_4 and D_4. Also B_3 extends in two ways, to B_4 and F_4. In fact, F_4 contains both B_4 and D_4, while H_4 contains A_4 and D_4.

The roots of D_4 may be taken as all 24 vectors of shape $(\pm1, \pm1, 0, 0)$. Or, on a different scale, the 8 vectors of shape $(\pm2, 0, 0, 0)$ together with the 16 of shape $(\pm1, \pm1, \pm1, \pm1)$.

The roots of F_4 are of two different lengths, and consist of the two copies of D_4 just mentioned. You can scale them so that either of them gives the short roots and the other one gives the long roots. To obtain B_4, take all of the long roots $(\pm1, \pm1, 0, 0)$ of F_4 and the eight short roots of shape $(\pm1, 0, 0, 0)$.

The roots of H_4 are those of D_4, best taken as $(\pm2, 0, 0, 0)$ and $(\pm1, \pm1, \pm1, \pm1)$, together with 96 more roots $(0, \pm1, \pm\sigma, \pm\tau)$, with any even permutation of the coordinates allowed. Inside here you can find A_4 consisting of 20 roots

$$(\pm2, 0, 0, 0), (0, \pm2, 0, 0), \pm(\pm1, \pm1, 1, 1), \pm(\pm1, 0, \tau, \sigma), \pm(0, \pm1, \sigma, \tau).$$

Reflections in quaternion notation are just the same as in the complex numbers. That is, reflection in 1 is the map

$$q \mapsto -\overline{q}$$

where $\overline{a+bi+cj+dk} = a-bi-cj-dk$. So reflection in r (assumed of norm 1) is the map

$$q \mapsto -r\overline{\overline{r}q} = -r\overline{q}r.$$

And the product of these two reflections is the rotation

$$q \mapsto rqr.$$

Scaling the roots of D_4 to norm 1 in the quaternions, we have $\pm 1, \pm i, \pm j, \pm k$ together with $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$. These units form a group variously known as $2.A_4$ or $SL_2(3)$ or the binary tetrahedral group. The first 8 of these form the quaternion group Q_8.

The full reflection group of F_4 has shape $2.(A_4 \times A_4).2.2$ in which the central 2 is negation of the whole 4-space, modulo which the two copies of A_4 are left- and right-multiplication by the units. Then the maps $q \mapsto \frac{1}{2}(1+i)q(1+i)$ and $q \mapsto \overline{q}$ extend this to the whole group.

There is a similar description of the reflection group of type H_4. This time the roots form a group of order 120, which is a double cover of A_5, also known as $SL_2(5)$ or the binary icosahedral group. The full reflection group has shape $2.(A_5 \times A_5).2$ in which we see left- and right-multiplications by this group of units, together with the map $q \mapsto \overline{q}$ again.

A complex description of D_4 is as the vectors $(2r, 0), (0, 2r), (r_1, r_2)$, where r is a short root of B_2 and r_1, r_2 are long roots. Depending on our choice of basis for B_2, this gives either

$$(\pm 2, 0), (\pm 2i, 0), (0, \pm 2), (0, \pm 2i), (\pm 1 \pm i, \pm 1 \pm i)$$

or

$$(\pm 1 \pm i, 0), (0, \pm 1 \pm i), (\pm 1, \pm 1), (\pm 1, \pm i), (\pm i, \pm 1), (\pm i, \pm i).$$

5 Dimensions bigger than 4

It turns out that A_4, B_4 and D_4 extend indefinitely, to arbitrary dimensions, but F_4 and H_4 do not extend at all. Also, D_5 extends in a different way to E_6, E_7 and E_8, at which point this series stops.

It is significant (I think) that the various series of reflection groups stop in dimensions 2, 4, and 8, precisely the dimensions of the complex numbers, quaternions, and octonions (Cayley numbers).

6 An eightfold path to E_8

There are innumerable ways to make E_8. I shall describe just a few of my favourites.

1. From F_4: take 48 roots $(2r, 0)$ where r is a short root of F_4, together with $24 \times 8 = 192$ roots (r, qr) where r is a long root and $q \in Q_8$.

2. If F_4 is labelled so that $\pm 1 \pm i$ are short and ± 2, etc, are long, this gives

$$\begin{array}{rl} (\pm 2, \pm 2, 0, 0 \mid 0, 0, 0, 0) & 48 \\ (\pm 2, 0, 0, 0 \mid \pm 2, 0, 0, 0) & 64 \\ (\pm 1, \pm 1, \pm 1, \pm 1 \mid \pm 1, \pm 1, \pm 1, \pm 1) & 128 \end{array}$$

   where in the last case there must be an even number of minus signs. Clearly we now see more symmetry, fusing the first two orbits of roots.

3. We can twist this by changing sign on one coordinate, so that there is an odd number of minus signs instead.

4. If F_4 is labelled so that $\pm 1, \pm i$ etc are short, and $\pm 1 \pm i$ etc are long, we get

$$\begin{array}{rl} (\pm 2, 0, 0, 0 \mid 0, 0, 0, 0) & 16 \\ (\pm 1, \pm 1, \pm 1, \pm 1 \mid 0, 0, 0, 0) & 32 \\ (\pm 1, \pm 1, 0, 0 \mid \pm 1, \pm 1, 0, 0) & 6.4.2.4 = 192 \end{array}$$

   where in the last case the right hand pair of 1s can either be in the same positions as the left hand pair, or in the complementary positions.

5. Labelling the coordinates $\infty, 0, 1, 3, 2, 6, 4, 5$ in that order enables us to describe the supports of the vectors of shape $(\pm 1^4, 0^4)$: they are either ∞ with a line $t, t+1, t+3 \pmod 7$ of the projective plane of order 2, or the complement thereof.

6. From B_2, take 16 roots $(2r, 0, 0, 0)$ where r is a short root of B_2, and $6.4.4 = 96$ roots $(r_1, r_2, 0, 0)$ where r_i are long roots of B_2, and 128 roots (r_1, r_2, r_3, r_4) where the r_i are short roots of B_2 and their product is ± 1.

7. Or twist this so that the product is $\pm i$ instead.

8. From H_4: define $N : \mathbb{Q}[\sqrt{5}] \to \mathbb{Q}$ by $N(a + b\sigma) = a$, and define a new norm $N(q\overline{q})$ on H_4. Now the things of norm 1 are the original things of norm 1, together with their multiples by σ. Thus we obtain 240 roots.

   (If we apply this process to H_2 we get A_4, and if we apply it to H_3 we get D_6.)

7 Properties of E_8

Reducing modulo 2 the lattice $\Lambda = \mathbb{Z}[\text{roots}] \cong \mathbb{Z}^8$ (as additive groups) gives $\Lambda/2\Lambda \cong (\mathbb{Z}/2\mathbb{Z})^8$. The Euclidean norm (on a suitable scale) reduces to a quadratic form mod 2, such that the 240 roots become the 120 vectors of norm 1. The reflections become orthogonal transvections, and generate the orthogonal group $O_8^+(2) = \Omega_8^+(2).2$ (or, in Atlas notation $O_8^+(2)$). Thus the reflection group modulo $\{\pm 1\}$ is this orthogonal group.

The 135 **isotropic vectors** i.e. the non-zero vectors of norm 0 in $\Lambda/2\Lambda$ come from congruence classes (modulo 2Λ) of vectors of twice the norm of a root. There are 16 vectors in each class, forming a cross. The stabilizer of a cross is $2^7 S_8$ (even sign changes and all coordinate permutations) so we can calculate the order of the reflection group as $2^{14}.3^5.5^2.7$.

Self-duality On the scale where the roots have norm 2, all the inner products of roots are integers. Conversely, any vector which has integer inner products with all the roots is in Λ.

To prove this, take our roots to be $(\pm 1, \pm 1, 0^6)$ and $(\pm \frac{1}{2}^8)$ with even signs.

- Inner product with $(2, 0^7)$ implies all coordinates are in $\frac{1}{2}\mathbb{Z}$
- Inner product with $(1, 1, 0^6)$ implies either all coordinates are in $\mathbb{Z}$ or all coordinates are in $\mathbb{Z} + \frac{1}{2}$.
- Inner product with $(\frac{1}{2}^8)$ implies the sum of the coordinates is $0 \bmod 2$, which gives the sign condition.

In fact, this property characterizes E_8: it is the unique self-dual even (i.e. all norms are even integers) integral (i.e. all inner products are integers) lattice in eight dimensions.