A new approach to the Leech lattice

Robert A. Wilson

Queen Mary, University of London

University of Cambridge, 21st October 2009

INTRODUCTION

The Leech lattice

The Leech lattice is a 24-dimensional lattice (i.e. discrete additive subgroup of $\mathbb{R}^{24}$) with many remarkable properties.

The Leech lattice

The Leech lattice is a 24-dimensional lattice (i.e. discrete additive subgroup of $\mathbb{R}^{24}$) with many remarkable properties.

- It is the unique even self-dual 24-dimensional lattice with no roots (i.e. vectors of norm 2).

The Leech lattice

The Leech lattice is a 24-dimensional lattice (i.e. discrete additive subgroup of $\mathbb{R}^{24}$) with many remarkable properties.

- It is the unique even self-dual 24-dimensional lattice with no roots (i.e. vectors of norm 2).
- Its 196560 minimal vectors (of norm 4) describe the unique way to pack 196560 (the maximum possible) unit spheres, all touching a given unit sphere.

The Leech lattice

The Leech lattice is a 24-dimensional lattice (i.e. discrete additive subgroup of $\mathbb{R}^{24}$) with many remarkable properties.

- It is the unique even self-dual 24-dimensional lattice with no roots (i.e. vectors of norm 2).
- Its 196560 minimal vectors (of norm 4) describe the unique way to pack 196560 (the maximum possible) unit spheres, all touching a given unit sphere.
- Its automorphism group is (a double cover of) Conway's sporadic simple group.

Constructing the Leech lattice

- The standard construction uses the (binary) Golay code, the unique (linear) perfect 3-error-correcting code of any length (or rather the extended code).

Constructing the Leech lattice

- The standard construction uses the (binary) Golay code, the unique (linear) perfect 3-error-correcting code of any length (or rather the extended code).
- There is also a construction in 12 complex dimensions, using the ternary Golay code,

Constructing the Leech lattice

- The standard construction uses the (binary) Golay code, the unique (linear) perfect 3-error-correcting code of any length (or rather the extended code).
- There is also a construction in 12 complex dimensions, using the ternary Golay code,
- and a construction in 6 quaternionic dimensions, using the hexacode, a certain code over the field of order 4.

Constructing the Leech lattice

- The standard construction uses the (binary) Golay code, the unique (linear) perfect 3-error-correcting code of any length (or rather the extended code).
- There is also a construction in 12 complex dimensions, using the ternary Golay code,
- and a construction in 6 quaternionic dimensions, using the hexacode, a certain code over the field of order 4.
- Why not 3 octonionic dimensions?!

Constructing the Leech lattice

- The standard construction uses the (binary) Golay code, the unique (linear) perfect 3-error-correcting code of any length (or rather the extended code).
- There is also a construction in 12 complex dimensions, using the ternary Golay code,
- and a construction in 6 quaternionic dimensions, using the hexacode, a certain code over the field of order 4.
- Why not 3 octonionic dimensions?!
- Several attempts have been made by several people over several decades to find such a construction, without any real success—until now.

PRELIMINARIES:

OCTONIONS AND E_8

Octonions

The (real) octonion algebra $\mathbb{O}$ is an 8-dimensional (non-associative) algebra

Octonions

The (real) octonion algebra $\mathbb{O}$ is an 8-dimensional (non-associative) algebra

- with an orthonormal basis $\{1 = i_\infty, i_0, \ldots, i_6\}$ labelled by the projective line $PL(7) = \{\infty\} \cup \mathbb{F}_7$,

Octonions

The (real) octonion algebra $\mathbb{O}$ is an 8-dimensional (non-associative) algebra

- with an orthonormal basis $\{1 = i_\infty, i_0, \ldots, i_6\}$ labelled by the projective line $PL(7) = \{\infty\} \cup \mathbb{F}_7$,
- with product given by $i_0 i_1 = -i_1 i_0 = i_3$ and images under the subscript permutations $t \mapsto t + 1$ and $t \mapsto 2t$.

Octonions

The (real) octonion algebra $\mathbb{O}$ is an 8-dimensional (non-associative) algebra

- with an orthonormal basis $\{1 = i_\infty, i_0, \ldots, i_6\}$ labelled by the projective line $PL(7) = \{\infty\} \cup \mathbb{F}_7$,
- with product given by $i_0 i_1 = -i_1 i_0 = i_3$ and images under the subscript permutations $t \mapsto t+1$ and $t \mapsto 2t$.
- The norm is $N(x) = x\overline{x}$, where $\overline{x}$ denotes the octonion conjugate of x, and satisfies $N(xy) = N(x)N(y)$.

Octonions

The (real) octonion algebra $\mathbb{O}$ is an 8-dimensional (non-associative) algebra

- with an orthonormal basis $\{1 = i_\infty, i_0, \ldots, i_6\}$ labelled by the projective line $PL(7) = \{\infty\} \cup \mathbb{F}_7$,
- with product given by $i_0 i_1 = -i_1 i_0 = i_3$ and images under the subscript permutations $t \mapsto t+1$ and $t \mapsto 2t$.
- The norm is $N(x) = x\overline{x}$, where $\overline{x}$ denotes the octonion conjugate of x, and satisfies $N(xy) = N(x)N(y)$.
- The Moufang laws hold in the octonions: $(xy)(zx) = x(yz)x$, $x(y(xz)) = (xyx)z$ and $((yx)z)x = y(xzx)$.

E_8

The E_8 root system embeds in this algebra in various interesting ways. For example,

E_8

The E_8 root system embeds in this algebra in various interesting ways. For example,

- take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and

E_8

The E_8 root system embeds in this algebra in various interesting ways. For example,

- take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and
- the 128 octonions $\frac{1}{2}(\pm \mathbf{1} \pm i_0 \pm \cdots \pm i_6)$ which have an odd number of minus signs.

E_8

The E_8 root system embeds in this algebra in various interesting ways. For example,

- take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and
- the 128 octonions $\frac{1}{2}(\pm \mathbf{1} \pm i_0 \pm \cdots \pm i_6)$ which have an odd number of minus signs.
- Denote by L the lattice spanned by these 240 octonions,

E_8

The E_8 root system embeds in this algebra in various interesting ways. For example,

- take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and
- the 128 octonions $\frac{1}{2}(\pm\mathbf{1} \pm i_0 \pm \cdots \pm i_6)$ which have an odd number of minus signs.
- Denote by L the lattice spanned by these 240 octonions,
- and write R for $\overline{L}$.

E_8

The E_8 root system embeds in this algebra in various interesting ways. For example,

- take the 240 roots to be the 112 octonions $\pm i_t \pm i_u$ for any distinct $t, u \in PL(7)$, and
- the 128 octonions $\frac{1}{2}(\pm 1 \pm i_0 \pm \cdots \pm i_6)$ which have an odd number of minus signs.
- Denote by L the lattice spanned by these 240 octonions,
- and write R for $\overline{L}$.
- Let $s = \frac{1}{2}(-1 + i_0 + \cdots + i_6)$, so that $s \in L$ and $\overline{s} \in R$.

Integral octonions

- $A_0 := \frac{1}{2}(1 + i_0)L = \frac{1}{2}R(1 + i_0)$ is closed under multiplication, and forms a copy of the Coxeter–Dickson integral octonions.

Integral octonions

- $A_0 := \frac{1}{2}(1 + i_0)L = \frac{1}{2}R(1 + i_0)$ is closed under multiplication, and forms a copy of the Coxeter–Dickson integral octonions.
- $L = (1 + i_0)A_0$ and $R = A_0(1 + i_0)$.

Integral octonions

- $A_0 := \frac{1}{2}(1 + i_0)L = \frac{1}{2}R(1 + i_0)$ is closed under multiplication, and forms a copy of the Coxeter–Dickson integral octonions.
- $L = (1 + i_0)A_0$ and $R = A_0(1 + i_0)$.
- It follows immediately from the Moufang law $(xy)(zx) = x(yz)x$ that $LR = (1 + i_0)A_0(1 + i_0)$.

Integral octonions

- $A_0 := \frac{1}{2}(1+i_0)L = \frac{1}{2}R(1+i_0)$ is closed under multiplication, and forms a copy of the Coxeter–Dickson integral octonions.
- $L = (1+i_0)A_0$ and $R = A_0(1+i_0)$.
- It follows immediately from the Moufang law $(xy)(zx) = x(yz)x$ that $LR = (1+i_0)A_0(1+i_0)$.
- Hence $B := \frac{1}{2}(1+i_0)A_0(1+i_0)$ satisfies

$$\begin{aligned} LR &= 2B \\ BL &= L \\ RB &= R \end{aligned}$$

Properties of L,R and B

- $L\overline{s} = 2B$.

Properties of L,R and B

- $L\overline{s} = 2B$.
- More generally, if ρ is any root in R then $L\rho = 2B$.

Properties of L,R and B

- $L\overline{s} = 2B$.
- More generally, if ρ is any root in R then $L\rho = 2B$.
- $2L \subset L\overline{s} \subset L$, and therefore $2L \subset Ls \subset L$.

Properties of L,R and B

- $L\overline{s} = 2B$.
- More generally, if ρ is any root in R then $L\rho = 2B$.
- $2L \subset L\overline{s} \subset L$, and therefore $2L \subset Ls \subset L$.
- $L\overline{s} + Ls = L$, so by self-duality of L we have $L\overline{s} \cap Ls = 2L$.

DEFINITIONS:

THE LEECH LATTICE
AND
THE CONWAY GROUP

The octonion Leech lattice

The octonionic Leech lattice $\Lambda = \Lambda_{\mathbb{O}}$ is the set of triples (x, y, z) of octonions, with norm $N(x, y, z) = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z})$, such that

The octonion Leech lattice

The octonionic Leech lattice $\Lambda = \Lambda_{\mathbb{O}}$ is the set of triples (x, y, z) of octonions, with norm $N(x, y, z) = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z})$, such that

1. $x, y, z \in L$;

The octonion Leech lattice

The octonionic Leech lattice $\Lambda = \Lambda_{\mathbb{O}}$ is the set of triples (x, y, z) of octonions, with norm $N(x, y, z) = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z})$, such that

1. $x, y, z \in L$;
2. $x + y, x + z, y + z \in L\overline{s}$;

The octonion Leech lattice

The octonionic Leech lattice $\Lambda = \Lambda_{\mathbb{O}}$ is the set of triples (x, y, z) of octonions, with norm $N(x, y, z) = \frac{1}{2}(x\overline{x} + y\overline{y} + z\overline{z})$, such that

1. $x, y, z \in L$;
2. $x + y, x + z, y + z \in L\overline{s}$;
3. $x + y + z \in Ls$.

The minimal vectors

The minimal vectors of Λ are the following 196560 vectors of norm 4, where λ is a root of L and $j, k \in J = \{\pm i_t \mid t \in PL(7)\}$:

Vectors		Number
$(2\lambda, 0, 0)$	$3 \times 240 =$	720
$(\lambda\bar{s}, \pm(\lambda\bar{s})j, 0)$	$3 \times 240 \times 16 =$	11520
$((\lambda s)j, \pm\lambda k, \pm(\lambda j)k)$	$3 \times 240 \times 16 \times 16 =$	184320
	Total =	196560

The octonionic Conway group

The following maps are symmetries of the octonionic Leech lattice, and generate the double cover $2{\cdot}Co_1$ of Conway's group:

The octonionic Conway group

The following maps are symmetries of the octonionic Leech lattice, and generate the double cover $2{\cdot}Co_1$ of Conway's group:

- Coordinate permutations

The octonionic Conway group

The following maps are symmetries of the octonionic Leech lattice, and generate the double cover $2{\cdot}Co_1$ of Conway's group:

- Coordinate permutations
- $r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$

The octonionic Conway group

The following maps are symmetries of the octonionic Leech lattice, and generate the double cover $2{\cdot}Co_1$ of Conway's group:

- Coordinate permutations
- $r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$
- $\frac{1}{2}R_{1-i_0}R_{1+i_t} : (x, y, z) \mapsto$
  $\frac{1}{2}((x(1-i_0))(1+i_t), (y(1-i_0))(1+i_t), (z(1-i_0))(1+i_t))$

The octonionic Conway group

The following maps are symmetries of the octonionic Leech lattice, and generate the double cover $2{\cdot}Co_1$ of Conway's group:

- Coordinate permutations
- $r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$
- $\frac{1}{2}R_{1-i_0}R_{1+i_t} : (x, y, z) \mapsto$ $\frac{1}{2}((x(1-i_0))(1+i_t), (y(1-i_0))(1+i_t), (z(1-i_0))(1+i_t))$
- The matrix

$$-\frac{1}{2}\begin{pmatrix} 0 & \overline{s} & \overline{s} \\ s & -1 & 1 \\ s & 1 & -1 \end{pmatrix}$$

interpreted as the map

$$(x, y, z) \mapsto -\frac{1}{2}((y+z)s, x\overline{s} - y + z, x\overline{s} + y - z).$$

Remarks

- The above construction is deceptively simple. In fact, however, finding the correct definition was not at all easy.

Remarks

- The above construction is deceptively simple. In fact, however, finding the correct definition was not at all easy.
- I was surely not the first person to notice, around 1980, that the number of minimal vectors in the Leech lattice is

$$196560 = 3 \times 240 \times (1 + 16 + 16 \times 16).$$

Remarks

- The above construction is deceptively simple. In fact, however, finding the correct definition was not at all easy.
- I was surely not the first person to notice, around 1980, that the number of minimal vectors in the Leech lattice is

$$196560 = 3 \times 240 \times (1 + 16 + 16 \times 16).$$

- I was surely not the first person, therefore, to try to build the Leech lattice from triples of integral octonions.

Remarks

- The above construction is deceptively simple. In fact, however, finding the correct definition was not at all easy.
- I was surely not the first person to notice, around 1980, that the number of minimal vectors in the Leech lattice is

$$196560 = 3 \times 240 \times (1 + 16 + 16 \times 16).$$

- I was surely not the first person, therefore, to try to build the Leech lattice from triples of integral octonions.
- But I believe I am the first person to provide a convincing explanation for this numerology.

Remarks

- The above construction is deceptively simple. In fact, however, finding the correct definition was not at all easy.
- I was surely not the first person to notice, around 1980, that the number of minimal vectors in the Leech lattice is

$$196560 = 3 \times 240 \times (1 + 16 + 16 \times 16).$$

- I was surely not the first person, therefore, to try to build the Leech lattice from triples of integral octonions.
- But I believe I am the first person to provide a convincing explanation for this numerology.
- One can also give nice descriptions of many of the maximal subgroups, for example the Suzuki chain subgroups.

SOME PROOFS, I:

THIS IS THE LEECH LATTICE

Some symmetries

- The definition of Λ is (obviously) invariant under permutations of the three coordinates, and under sign-changes on any coordinates.

Some symmetries

- The definition of Λ is (obviously) invariant under permutations of the three coordinates, and under sign-changes on any coordinates.
- It is also invariant under the map

$$r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$$

Some symmetries

- The definition of Λ is (obviously) invariant under permutations of the three coordinates, and under sign-changes on any coordinates.
- It is also invariant under the map

$$r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$$

- Certainly $Li_t = L$, so the first condition of the definition is preserved.

Some symmetries

- The definition of Λ is (obviously) invariant under permutations of the three coordinates, and under sign-changes on any coordinates.
- It is also invariant under the map

$$r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$$

- Certainly $Li_t = L$, so the first condition of the definition is preserved.
- Then $y(1 - i_t) \in LR = 2B = L\bar{s}$, so the second condition is preserved.

Some symmetries

- The definition of Λ is (obviously) invariant under permutations of the three coordinates, and under sign-changes on any coordinates.
- It is also invariant under the map

$$r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$$

- Certainly $Li_t = L$, so the first condition of the definition is preserved.
- Then $y(1 - i_t) \in LR = 2B = L\bar{s}$, so the second condition is preserved.
- Finally $(y + z)(1 - i_t) \in 2BL = 2L \subset Ls$, so the third condition is preserved also.

Some vectors in the Leech lattice

Suppose that λ is any root in L.

Some vectors in the Leech lattice

Suppose that λ is any root in L.

- Then the vector $(\lambda s, \lambda, -\lambda)$ lies in Λ, since $Ls \subseteq L$ and $\lambda s + \lambda = \lambda(s+1) = -\lambda\bar{s}$.

Some vectors in the Leech lattice

Suppose that λ is any root in L.

- Then the vector $(\lambda s, \lambda, -\lambda)$ lies in Λ, since $Ls \subseteq L$ and $\lambda s + \lambda = \lambda(s+1) = -\lambda\bar{s}$.
- Therefore Λ contains the vectors $(\lambda s, \lambda, \lambda) + (\lambda, \lambda s, -\lambda) = -(\lambda\bar{s}, \lambda\bar{s}, 0)$, that is, all vectors $(2\beta, 2\beta, 0)$ with β a root in $\frac{1}{2}L\bar{s} = B$.

Some vectors in the Leech lattice

Suppose that λ is any root in L.

- Then the vector $(\lambda s, \lambda, -\lambda)$ lies in Λ, since $Ls \subseteq L$ and $\lambda s + \lambda = \lambda(s+1) = -\lambda\bar{s}$.
- Therefore Λ contains the vectors $(\lambda s, \lambda, \lambda) + (\lambda, \lambda s, -\lambda) = -(\lambda\bar{s}, \lambda\bar{s}, 0)$, that is, all vectors $(2\beta, 2\beta, 0)$ with β a root in $\frac{1}{2}L\bar{s} = B$.
- Hence Λ also contains

$$(\lambda(1+i_0), \lambda(1+i_0), 0) + (\lambda(1-i_0), -\lambda(1+i_0), 0) = (2\lambda, 0, 0).$$

The minimal vectors

Applying the above symmetries, Λ contains the following 196560 vectors of norm 4, where λ is a root of L and $j, k \in J = \{\pm i_t \mid t \in PL(7)\}$:

Vectors		Number
$(2\lambda, 0, 0)$	$3 \times 240 =$	720
$(\lambda\bar{s}, \pm(\lambda\bar{s})j, 0)$	$3 \times 240 \times 16 =$	11520
$((\lambda s)j, \pm\lambda k, \pm(\lambda j)k)$	$3 \times 240 \times 16 \times 16 =$	184320
	Total =	196560

Identification with the real Leech lattice

I claim that the 196560 vectors listed above span Λ.

Identification with the real Leech lattice

I claim that the 196560 vectors listed above span Λ.

- For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0.

Identification with the real Leech lattice

I claim that the 196560 vectors listed above span Λ.

- For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0.
- Then we know that $y \in L\bar{s}$, so by adding suitable vectors of the second type we may reduce y to 0 also.

Identification with the real Leech lattice

I claim that the 196560 vectors listed above span Λ.

- For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0.
- Then we know that $y \in L\bar{s}$, so by adding suitable vectors of the second type we may reduce y to 0 also.
- Finally we have that $x \in L\bar{s} \cap Ls = 2L$ so we can reduce x to 0 also.

Identification with the real Leech lattice

I claim that the 196560 vectors listed above span Λ.

- For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0.
- Then we know that $y \in L\bar{s}$, so by adding suitable vectors of the second type we may reduce y to 0 also.
- Finally we have that $x \in L\bar{s} \cap Ls = 2L$ so we can reduce x to 0 also.
- Thus the claim is proved.

Identification with the real Leech lattice

I claim that the 196560 vectors listed above span Λ.

- For if $(x, y, z) \in \Lambda$, then by adding suitable vectors of the third type, we may reduce z to 0.
- Then we know that $y \in L\bar{s}$, so by adding suitable vectors of the second type we may reduce y to 0 also.
- Finally we have that $x \in L\bar{s} \cap Ls = 2L$ so we can reduce x to 0 also.
- Thus the claim is proved.

At this stage it is easy to identify Λ with the Leech lattice in a number of different ways.

The MOG labelling

- Label the coordinates of each brick of the MOG as follows:

$$\frac{1}{2}\begin{array}{|cc|}\hline -1 & i_0 \\ i_4 & i_5 \\ i_2 & i_6 \\ i_1 & i_3 \\ \hline\end{array}$$

The MOG labelling

- Label the coordinates of each brick of the MOG as follows:

$$\frac{1}{2}\begin{array}{|cc|} \hline -1 & i_0 \\ i_4 & i_5 \\ i_2 & i_6 \\ i_1 & i_3 \\ \hline \end{array}$$

- the vectors $(1-i_0)(s,-1,-1)$ and $s(-s,1,1)$ are then

	2	2	2	2	2
2					
2					
2					

−3	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

...continued

- These vectors are in the standard Leech lattice, and in the octonionic Leech lattice.

...continued

- These vectors are in the standard Leech lattice, and in the octonionic Leech lattice.
- Similarly and by symmetry, the vectors $(1 \pm i_t)(s, 1, 1)$ are in both Leech lattices.

...continued

- These vectors are in the standard Leech lattice, and in the octonionic Leech lattice.
- Similarly and by symmetry, the vectors $(1 \pm i_t)(s, 1, 1)$ are in both Leech lattices.
- But, as we have seen, these vectors, together with images under (octonionic) coordinate permutations and sign-changes, span the lattice.

The lattice is self-dual

An alternative approach is to show directly from our definition that Λ is an even self-dual lattice with no vectors of norm 2, whence it is the Leech lattice by Conway's characterisation.

SOME PROOFS, II:

THIS IS THE CONWAY GROUP

Some 'diagonal' symmetries

- Reflection in r (an octonion of norm 1) is the map $x \mapsto -r\overline{x}r$. In particular, $s = -\frac{1}{2}(1+i_t)\overline{s}(1+i_t)$.

Some 'diagonal' symmetries

- Reflection in r (an octonion of norm 1) is the map $x \mapsto -r\overline{x}r$. In particular, $s = -\frac{1}{2}(1+i_t)\overline{s}(1+i_t)$.
- Using R_a to denote right-multiplication by a, the Moufang law $R_aR_bR_a = R_{aba}$ implies

$$\begin{aligned} R_sR_{1-i_0}R_{1+i_t} &= R_{1+i_0}R_{1-i_t}R_s \\ R_{\overline{s}}R_{1-i_0}R_{1+i_t} &= R_{1+i_0}R_{1-i_t}R_{\overline{s}}. \end{aligned}$$

Some 'diagonal' symmetries

- Reflection in r (an octonion of norm 1) is the map $x \mapsto -r\overline{x}r$. In particular, $s = -\frac{1}{2}(1+i_t)\overline{s}(1+i_t)$.
- Using R_a to denote right-multiplication by a, the Moufang law $R_aR_bR_a = R_{aba}$ implies

$$\begin{aligned} R_sR_{1-i_0}R_{1+i_t} &= R_{1+i_0}R_{1-i_t}R_s \\ R_{\overline{s}}R_{1-i_0}R_{1+i_t} &= R_{1+i_0}R_{1-i_t}R_{\overline{s}}. \end{aligned}$$

- Therefore

$$\begin{aligned} (L(1-i_0))(1+i_t) &= (LR)L = 2BL = 2L \\ ((L\overline{s})(1-i_0))(1+i_t) &= ((L(1+i_0))(1-i_t))\overline{s} = 2L\overline{s} \\ ((Ls)(1-i_0))(1+i_t) &= ((L(1+i_0))(1-i_t))s = 2Ls. \end{aligned}$$

Some ‘diagonal’ symmetries

- Reflection in r (an octonion of norm 1) is the map $x \mapsto -r\overline{x}r$. In particular, $s = -\frac{1}{2}(1+i_t)\overline{s}(1+i_t)$.
- Using R_a to denote right-multiplication by a, the Moufang law $R_aR_bR_a = R_{aba}$ implies

$$\begin{array}{rcl} R_sR_{1-i_0}R_{1+i_t} & = & R_{1+i_0}R_{1-i_t}R_s \\ R_{\overline{s}}R_{1-i_0}R_{1+i_t} & = & R_{1+i_0}R_{1-i_t}R_{\overline{s}}. \end{array}$$

- Therefore

$$\begin{array}{rcl} (L(1-i_0))(1+i_t) & = & (LR)L = 2BL = 2L \\ ((L\overline{s})(1-i_0))(1+i_t) & = & ((L(1+i_0))(1-i_t))\overline{s} = 2L\overline{s} \\ ((Ls)(1-i_0))(1+i_t) & = & ((L(1+i_0))(1-i_t))s = 2Ls. \end{array}$$

- In other words the map $\frac{1}{2}R_{1-i_0}R_{1+i_t}$ preserves the octonion Leech lattice.

The monomial subgroup

These symmetries generate a double cover $2{\cdot}A_8$ of A_8.

The monomial subgroup

These symmetries generate a double cover $2{\cdot}A_8$ of A_8.

- For the roots $1 + i_t$ for $0 \leq t \leq 6$ form a copy of the root system of type A_7, whose Weyl group is the symmetric group S_8.

The monomial subgroup

These symmetries generate a double cover $2{\cdot}A_8$ of A_8.

- For the roots $1+i_t$ for $0 \le t \le 6$ form a copy of the root system of type A_7, whose Weyl group is the symmetric group S_8.
- The product of the reflections in $1+i_0$ and $1+i_t$ is the map $x \mapsto \frac{1}{4}(1+i_t)((1-i_0)x(1-i_0))(1+i_t)$ that is the product of two bi-multiplications $\frac{1}{2}B_{1-i_0}\frac{1}{2}B_{1+i_t}$.

The monomial subgroup

These symmetries generate a double cover $2{\cdot}A_8$ of A_8.

- For the roots $1 + i_t$ for $0 \leq t \leq 6$ form a copy of the root system of type A_7, whose Weyl group is the symmetric group S_8.
- The product of the reflections in $1 + i_0$ and $1 + i_t$ is the map $x \mapsto \frac{1}{4}(1 + i_t)((1 - i_0)x(1 - i_0))(1 + i_t)$ that is the product of two bi-multiplications $\frac{1}{2}B_{1-i_0}\frac{1}{2}B_{1+i_t}$.
- These elements generate the rotation part A_8 of the Weyl group, and applying the triality automorphism the maps $\frac{1}{2}R_{1-i_0}R_{1+i_t}$ generate $2A_8$.

The monomial subgroup

These symmetries generate a double cover $2{\cdot}A_8$ of A_8.

- For the roots $1 + i_t$ for $0 \leq t \leq 6$ form a copy of the root system of type A_7, whose Weyl group is the symmetric group S_8.
- The product of the reflections in $1 + i_0$ and $1 + i_t$ is the map $x \mapsto \frac{1}{4}(1 + i_t)((1 - i_0)x(1 - i_0))(1 + i_t)$ that is the product of two bi-multiplications $\frac{1}{2}B_{1-i_0}\frac{1}{2}B_{1+i_t}$.
- These elements generate the rotation part A_8 of the Weyl group, and applying the triality automorphism the maps $\frac{1}{2}R_{1-i_0}R_{1+i_t}$ generate $2A_8$.
- Adjoining coordinate permutations and sign changes we get a group $2A_8 \times S_4$.

The monomial subgroup

These symmetries generate a double cover $2{\cdot}A_8$ of A_8.

- For the roots $1 + i_t$ for $0 \le t \le 6$ form a copy of the root system of type A_7, whose Weyl group is the symmetric group S_8.
- The product of the reflections in $1 + i_0$ and $1 + i_t$ is the map $x \mapsto \frac{1}{4}(1 + i_t)((1 - i_0)x(1 - i_0))(1 + i_t)$ that is the product of two bi-multiplications $\frac{1}{2}B_{1-i_0}\frac{1}{2}B_{1+i_t}$.
- These elements generate the rotation part A_8 of the Weyl group, and applying the triality automorphism the maps $\frac{1}{2}R_{1-i_0}R_{1+i_t}$ generate $2A_8$.
- Adjoining coordinate permutations and sign changes we get a group $2A_8 \times S_4$.
- Adjoining the symmetry $r_t : (x, y, z) \mapsto (x, yi_t, zi_t)$, this extends to a group of shape $2^{3+12}(A_8 \times S_3)$.

The Suzuki chain

The so-called Suzuki chain of subgroups of $2{}^{\cdot}Co_1$ is a series of subgroups of the following shapes:

$$\begin{array}{rcl}
2{}^{\cdot}A_9 & \times & S_3 \\
2{}^{\cdot}A_8 & \times & S_4 \\
(2{}^{\cdot}A_7 & \times & L_3(2)):2 \\
(2{}^{\cdot}A_6 & \times & U_3(3)):2 \\
(2{}^{\cdot}A_5 & \circ & 2{}^{\cdot}J_2):2 \\
(2{}^{\cdot}A_4 & \circ & 2{}^{\cdot}G_2(4)):2 \\
 & & 6{}^{\cdot}Suz{:}2
\end{array}$$

Generators for the Conway group

- To obtain $2{\cdot}A_9 \times S_3$, take the S_3 of coordinate permutations, and $2{\cdot}A_8$, and extend $2{\cdot}A_7$ to $2{\cdot}S_7$ by adjoining the element $\frac{1}{2}R_{i_0-i_1}R_s^*$, where

$$R_s^* := R_s \frac{1}{2} \begin{pmatrix} s & 1 & 1 \\ 1 & s & 1 \\ 1 & 1 & s \end{pmatrix}.$$

Generators for the Conway group

- To obtain $2{\cdot}A_9 \times S_3$, take the S_3 of coordinate permutations, and $2{\cdot}A_8$, and extend $2{\cdot}A_7$ to $2{\cdot}S_7$ by adjoining the element $\frac{1}{2}R_{i_0-i_1}R_s^*$, where

$$R_s^* := R_s \frac{1}{2} \begin{pmatrix} s & 1 & 1 \\ 1 & s & 1 \\ 1 & 1 & s \end{pmatrix}.$$

- Since both this subgroup and the monomial subgroup are maximal in $2{\cdot}Co_1$, we now have generators for the Conway group.

Generators for the other Suzuki chain groups

- To obtain $(2{\cdot}A_7 \times L_3(2)).2$ take the subgroup $2{\cdot}A_7$ of $2{\cdot}A_8$ generated by $\frac{1}{2}R_{i_0-i_1}R_{i_0-i_t}$, together with the complex reflection group $2 \times L_3(2)$ generated by the monomial $2 \times S_4$ together with reflection in $(s, 1, 1)$, and adjoin $\frac{1}{2}R_{i_0-i_1}R_s^*$.

Generators for the other Suzuki chain groups

- To obtain $(2{\cdot}A_7 \times L_3(2)).2$ take the subgroup $2{\cdot}A_7$ of $2{\cdot}A_8$ generated by $\frac{1}{2}R_{i_0-i_1}R_{i_0-i_t}$, together with the complex reflection group $2 \times L_3(2)$ generated by the monomial $2 \times S_4$ together with reflection in $(s, 1, 1)$, and adjoin $\frac{1}{2}R_{i_0-i_1}R_s^*$.
- To obtain the remaining groups in the list, adjoin to $L_3(2)$ the $2{\cdot}A_{9-n}$ in $2{\cdot}A_9$ which commutes with the given $2{\cdot}A_n$.

The involution centraliser

- The involution centraliser has the shape $2^{1+8} \cdot W(\mathrm{E}_8)'$.

The involution centraliser

- The involution centraliser has the shape $2^{1+8}\cdot W(\mathrm{E}_8)'$.
- Take the involution $\mathrm{diag}(1,-1,-1)$. Then the normal subgroup 2^{1+8} is generated by $r_t = \mathrm{diag}(1, i_t, i_t)$ for all t.

The involution centraliser

- The involution centraliser has the shape $2^{1+8} \cdot W(\mathrm{E}_8)'$.
- Take the involution $\mathrm{diag}(1, -1, -1)$. Then the normal subgroup 2^{1+8} is generated by $r_t = \mathrm{diag}(1, i_t, i_t)$ for all t.
- Modulo this, the group $2 \cdot A_8$ together with $\mathrm{diag}(i_t, i_t, 1)$ generate a maximal subgroup of $W(\mathrm{E}_8)'$, which may be extended to the whole group by adjoining an element such as

$$\frac{1}{2} R_{1-i_0} \begin{pmatrix} s & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

NOT THE END

Applications

- Perhaps this gives us a better understanding of why the Leech lattice exists.

Applications

- Perhaps this gives us a better understanding of why the Leech lattice exists.
- Perhaps it will give us new ways to prove important properties of the Leech lattice and the Conway group.

Applications

- Perhaps this gives us a better understanding of why the Leech lattice exists.
- Perhaps it will give us new ways to prove important properties of the Leech lattice and the Conway group.
- Perhaps we can use octonions to simplify the construction of the Monster.

Applications

- Perhaps this gives us a better understanding of why the Leech lattice exists.
- Perhaps it will give us new ways to prove important properties of the Leech lattice and the Conway group.
- Perhaps we can use octonions to simplify the construction of the Monster.
- Perhaps it will explain the '2-local group' $BDI(4)$ which contains Co_3 and looks as though it should be some kind of twist of 'skew-symmetric 3×3 matrices over octonions'.

THE END