Finite simple groups **Problem sheet 1**

Exercise 1. *For a permutation $\pi \in S_n$ define*

$$\varepsilon(\pi) = \prod_{1 \le i < j \le n} \frac{i-j}{i^\pi - j^\pi} \in \mathbb{Q}.$$

Show that $\varepsilon = \pm 1$ and that ε is a group homomorphism from S_n onto $C_2 = \{1, -1\}$. Hence obtain another proof that the sign of a permutation is well-defined.

Exercise 2. *Let $G < S_n$ act transitively on $\Omega = \{1, \ldots, n\}$ and let $H = \{g \in G : a^g = a\}$ for fixed $a \in \Omega$. Prove that $\phi : a^g \mapsto Hg$ is a bijection between Ω and the set $G : H$ of right cosets of H in G.*

Prove also that $Hg = \{x \in G : a^x = a^g\}$.

Exercise 3. *Prove that the orbits of a group H acting on a set Ω form a partition of Ω.*

Exercise 4. *Show that A_n is not $(n-1)$-transitive.*

Exercise 5. *Let G act transitively on Ω. Show that the average number of fixed points of the elements of G is 1, i.e.*

$$\frac{1}{|G|} \sum_{g \in G} |\{x \in \Omega \mid x^g = x\}| = 1.$$

Exercise 6. *Verify that the semidirect product $G :_\phi H$. Show that the subset $\{(g, 1_H) : g \in G\}$ is a normal subgroup isomorphic to G, and that the subset $\{(1_G, h) : h \in H\}$ is a subgroup isomorphic to H.*

Exercise 7. *Suppose that G has a normal subgroup A and a subgroup B satisfying $G = AB$ and $A \cap B = 1$. Prove that $G \cong A{:}_\phi B$, where $\phi : B \to \mathrm{Aut} A$ is defined by $\phi(b) : a \mapsto b^{-1}ab$.*

Exercise 8. *Prove that if the permutation π on n points is the product of k disjoint cycles (including trivial cycles), then π is an even permutation if and only if $n - k$ is an even integer.*

Exercise 9. *Determine the number of conjugacy classes in A_8, and write down one element from each class.*

Exercise 10. *Show that if $n \ge 5$ then there is no non-trivial conjugacy class in A_n with fewer than n elements.*

EXERCISE 11. *Let S_5 act on the 10 unordered pairs $\{a,b\} \subset \{1,2,3,4,5\}$. Show that this action is primitive. Determine the stabilizer of one of the 10 pairs, and deduce that it is a maximal subgroup of S_5.*

EXERCISE 12. *The previous question defines a primitive embedding of S_5 in S_{10}. Show that this S_5 is not maximal in S_{10}.*

[Hint: construct a primitive action of S_6 on 10 points, extending this action of S_5.]

EXERCISE 13. *If $k < \frac{n}{2}$, show that the action of S_n on the $\binom{n}{k}$ unordered k-tuples is primitive.*

EXERCISE 14. *If G acts k-transitively on $\{1,2,\ldots,n\}$ for some $k > 1$, and H is the stabilizer of the point n, show that H acts $(k-1)$-transitively on $\{1,2,\ldots,n-1\}$.*

EXERCISE 15. *Let G be the group of permutations of 8 points $\{\infty,0,1,2,3,4,5,6\}$ generated by $(0,1,2,3,4,5,6)$ and $(1,2,4)(3,6,5)$ and $(\infty,0)(1,6)(2,3)(4,5)$. Show that G is 2-transitive. Show that the Sylow 7-subgroups of G have order 7, and that their normalisers have order 21. Show that there are just 8 Sylow 7-subgroups, and deduce that G has order 168. Show that G is simple.*

EXERCISE 16. *Let x be an element in S_n of cycle type $(c_1^{n_1},\ldots,c_k^{n_k})$, where $c_1,\ldots,c_k$ are distinct positive integers. Show that the centralizer of x in S_n has the shape $(C_{c_1} \wr S_{n_1}) \times \cdots \times (C_{c_k} \wr S_{n_k})$.*

EXERCISE 17. *Show that if $H \cong \mathrm{AGL}_3(2) \cong 2^3{:}\mathrm{GL}_3(2)$ is a subgroup of S_8, and $K = H^g$ where g is an odd permutation, then H and K are not conjugate in A_8.*

EXERCISE 18. *Prove that $S_k \wr S_2$ is maximal in S_{2k} for all $k \geq 2$.*

EXERCISE 19. *Prove that $S_k \wr S_m$ is maximal in S_{km} for all $k,m \geq 2$.*

EXERCISE 20. *Prove that the 'diagonal' subgroups of S_n (as defined in the notes) are primitive.*

EXERCISE 21. *Show that if H is abelian and transitive on Ω, then it is regular on Ω.*

EXERCISE 22. *Use the O'Nan–Scott theorem to write down as many maximal subgroups of S_5 as you can. Can you prove your subgroups are maximal?*

EXERCISE 23. *Do the same for A_5.*