Embeddings of $Sz(32)$ in $E_8(5)$

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Abstract

We show that the Suzuki group, $Sz(32)$, is a subgroup of $E_8(5)$, and so is its automorphism group. Both are unique up to conjugacy in $E_8(\mathbb{F})$ for any field $\mathbb{F}$ of characteristic 5, and the automorphism group $Sz(32):5$ is maximal in $E_8(5)$.

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1 Introduction

Finite subgroups of simple exceptional Lie groups have received much attention recently. There are many interesting examples—see [4], [5], [7], [8], [11], [12], [13] and [22]. If one allows the exceptional Lie group (over $\mathbb{F} = \mathbb{C}$) to be replaced by an exceptional simple algebraic group over a field $\mathbb{F}$ of positive characteristic, these plus further interesting embeddings occur. For example, there is an embedding of $L_4(5)$ in $E_8(\mathbb{F})$ precisely when $\mathbb{F}$ contains the Galois field $\mathbb{F}_4$ [6]. There are further interesting examples in [16] and [17], where the embeddings of sporadic simple groups are determined. More recently, Liebeck and Seitz [19] have considered the question of cross-characteristic embeddings of groups of Lie type in exceptional groups—see the survey paper of Liebeck [18], where these and other related results are discussed. In this paper we resolve one of the last two cases in [19] to be settled. In particular we show there is an embedding of the Suzuki group $Sz(32) = {}^2B_2(32)$ into $E_8(\mathbb{F})$ when $\mathbb{F}$ has characteristic 5.

Theorem 1 *There is a unique conjugacy class of subgroups $Sz(32)$ in $E_8(\mathbb{F})$ where $\mathbb{F}$ is a field of characteristic 5. The normalizer of this in $E_8(\mathbb{F})$ is the automorphism group $Sz(32){:}5$ of $Sz(32)$. They both act on the 248-dimensional module as a non-splitting extension of the two distinct 124-dimensional irreducible modules. The group $Sz(32){:}5$ is a maximal subgroup of $E_8(5)$. The group $Sz(32)$ does not embed in $E_8(\mathbb{F})$ for fields of characteristic not 5 or 2.*

The fact that there are no embeddings in other characteristics (other than 2) was established in [4] in characteristic 0 and in [19] for positive characteristics. We outline some of the arguments below for completeness. Our construction uses an embedding of the Borel subgroup $2^{5+5}{:}31$ of $Sz(32)$ into the well-known 2-local subgroup $2^{5+10}L_5(2)$ of E_8. We remark that the resulting subgroup $Sz(32){:}5$ is a maximal subgroup of $E_8(5)$ using [19]: If X were a proper subgroup of $E_8(5)$ containing $Sz(32):5$ properly, it would have to be simple, and in fact classical of characteristic 5; on the other hand, $Sz(32)$ has no nontrivial representations in characteristic other than 2 of dimension less than 124.

2 Character considerations

Let $E = E_8(\mathbb{F}),\ S = Sz(32)$. Here we assume $\mathbb{F}$ is a field of characteristic not 2 and assume S is a subgroup of E. By considering the ordinary and modular character tables of S, we see that the 248-dimensional adjoint module V for E must have the two irreducible constituents of dimension 124 when restricted to S. This follows as V is self dual, and the only non-trivial irreducible representation degrees in characteristic not 2 of dimension ≤ 248 are the two of dimension 124. Let W be a module for S of dimension 248 with character the sum of the two distinct irreducibles χ_1, χ_2 of dimension 124.

We include an argument to show an embedding of $Sz(32)$ in $E_8(\mathbb{F})$ can occur only for $\mathbb{F}$ of characteristic 5 for completeness. It is also shown in [19]. Consider the skew symmetric square $\Lambda^2 W$ of W. Notice, using [9], [15], or using GAP, that neither χ_1 nor χ_2 appear as constituents of the character of this, except in characteristic 5. Indeed this follows from considerations of the Brauer tree and the fact that in characteristic 0, the skew symmetric square is a sum of the trivial character, the fifteen characters of degree 1025, and twice the six characters of degree 1271. In characteristic 5 only, these last characters contain as constituents the irreducible characters of degree 124. This means that there is no embedding of S in E except possibly in characteristic 5, since the existence of the Lie product and the Killing form implies that $\Lambda^2 V$ has a submodule isomorphic to V.

Thus from now on we assume that $\mathbb{F}$ has characteristic 5. Using the theory of cyclic defect groups, we see that there are two possibilities at this stage: either W is a direct sum of the two irreducible constituents, or it is a nonsplitting extension. In this latter case, the order in which the two constituents appear is determined by the Brauer tree. For a discussion of this and the Brauer tree see [1].

3 The Borel of $Sz(32)$ in E

The Borel subgroup, B, of $S = Sz(32)$ is a subroup of type $2^{5+5}{:}31$. The Sylow 2 subgroup has centre, K, of order 32 of type 2^5. The Sylow 2 subgroup is P of type 2^{5+5} and all elements in $P \setminus K$ have order 4. Moreover, once the action of the element of order 31 is determined on K, the action on P/K is also determined.

Lemma 1 *There is at most one conjugacy class of subgroups isomorphic to B in $E_8(\mathbb{F})$, with $\mathbb{F}$ of characteristic* 5.

Proof. We remark that the existence of such subgroups is not required in the proof of Theorem 1. However, this follows from Theorem 1, and is also proved in Section 8.

Now B has an elementary abelian normal subgroup of order 32, and an element of order 31 normalizing it. As in [6], Lemma 2.17, we see that there is a unique such subgroup of type $2^5{:}31$ in E. This means that any embedding of B in E can be conjugated in E so that this elementary subgroup 2^5 is a particular conjugate, K, and our group B lies in the normalizer of K in E.

It is well known (see [2], [3] or [6]) that E has a subgroup $M = 2^{5+10}L_5(2)$, and there is a unique conjugacy class of such subgroups. Each is the full normalizer of an elementary abelian group K of order 32, all of whose involutions are conjugate and have trace -8 when acting on V. We may assume then by the above paragraph that after conjugation, B is a subgroup of M, and both B and M normalize K.

We write $Q = O_2(M)$. By [6], the module Q/K for $L_5(2)$ is the skewsymmetric square of a natural 5-dimensional module. This implies that an element of order 31 has two distinct irreducibles of dimension 5 when acting on Q/K. Thus M contains just two classes of subgroups of type 2^{5+5}:31 in which the group 2^5 is K. Moreover, these are not isomorphic to each other, and therefore at most one of them is isomorphic to the Borel subgroup of $Sz(32)$. This completes the proof.

4 Smith's construction of the Dempwolff group and generating matrices

In his thesis [23], Peter Smith constructed a certain subgroup D of $E_8(\mathbb{C})$ called the Dempwolff group. Here D is a nonsplit extension of 2^5 by $L_5(2)$, which was then used by Thompson to construct the Thompson sporadic simple group Th. He produced 248×248 matrices which generate D and preserve the natural Lie algebra structure. These matrices have rational entries with denominators powers of 2. By reduction mod 5, these lead to an embedding of D in E. This group D is a maximal subgroup of the group M as described in the section above. The subgroups of type 2^5:31 are conjugate to the groups as described above, and the subgroup of order 32 can be taken to be K.

We now produce matrices that generate M. In this section, we give an overview of the method, and give more specific details in the later sections. We take the matrices given by Peter Smith that generate D. In his construction, a subgroup of K of index 2 lies in a split torus T, and an element z outside is an involution corresponding to the centre of the Weyl group, inverting the torus. We found an involution t centralizing K which is in $T \setminus K$, of the form $h_\alpha(-1)$, with α a fundamental root. Recall the torus is spanned by elements in E of the form $h_\alpha(\lambda)$ for λ in $\mathbb{F}$. When $\lambda = -1$ this has order 2 and commutes with z. Computations as described in the next section give an element π in D of order 31. Taking t with this element π generates the group $2^{5+10} : 31$. This follows as it contains an involution outside K. Using the fact that the element π has two distinct minimal polynomials on the constituents of Q/K, we can obtain the group P of type 2^{5+5}. Now taking the group generated by this together with our element π gives a group B isomorphic to the Borel subgroup of S which is in E and is uniquely determined up to conjugacy. These are all over the field with 5 elements.

What is needed now is another element in S, not in B, which we take to be an involution v inverting π. We find one over the field with 5 elements. This was found separately by computer as described below. We first found a suitable nonsplit extension, V, of dimension 248, for S, with the right constituents for S. Now V has an S-submodule V_1, for which B acts irreducibly and differently on V_1 and V/V_1. We are able to do this so that the elements of B are in E_8. The extension to S by adjoining v fixes V and so acts on V/V_1. The action is unique on the irreducible constituents, because B acts irreducibly. In terms of matrices, if we write the representation with respect to a basis obtained by extending a basis for the 124-dimensional submodule, then the lower left block of the matrices is non-zero; there are five different extensions, obtained by multiplying this lower left block by scalars. Multiplying by 0, for instance, gives a direct sum. In each case, these are in $GL_{248}(5)$. We found that precisely one of the choices gave an element v of order 2 which could possibly be in E. We determined this first by checking orders of words in v and other generators. These orders did not divide the order of E except in the one, nonzero, case. We later showed directly that these other cases were not in E. The next sections describe in some detail how these calculations were done.

5 5-modular representations of $Sz(32)$

All of our work is with the field with 5 elements only. Because of lemma 1, and using [19] this is sufficient for theorem 1. In particular, after conjugation, any two subgroups isomorphic to S can be assumed to have the same Borel subgroup B. Then any extension by different involutions must fix the same invariant series, and by [19] can only be S. Because the Sylow 5-subgroup of $Sz(32)$ is cyclic of order 25, the 5-modular representation theory is completely determined by the Brauer tree, which is easy to calculate from the ordinary character table [9] (see also [15]). In particular, using the theory of cyclic blocks (see Chapter 5 of [1]) the extensions of one 124-dimensional irreducible by the other form a one-dimensional space. The zero element of this space corresponds to the direct sum, while the non-zero elements correspond to four equivalent non-split extensions.

To make these 248-dimensional modules explicitly, we use the Meat-axe [21], including a variant of the condensation method [20].

We begin with the 124-dimensional module over $GF(41)$ which is in the world-wide-web group atlas [24], and can itself be obtained as a constituent of the 'natural' permutation module on 1025 points. In the group, we find a maximal subgroup 41:4, and then find a 1-space which is invariant under this subgroup. We then make the permutation action of $Sz(32)$ on the 198400 images of this 1-space.

In practice, we take the 'standard' generators a, b for $Sz(32)$ as defined in [24], which are defined by taking a of order 2, b of order 4, with ab of order 5, $abab^2$ of order 25, and $ab(abab^2)^2ab^2$ also of order 25. It is quite straightforward to check computationally that such a pair of generators is unique up to automorphisms. A simple search will now produce generators for a subgroup 41:4, such as $y = (abab^2)^{-5}b(abab^2)^5$ of order 4, and $x = ay^2$ of order 41.

6 Condensation

Next we 'condense' this permutation module modulo 5 in the following way. As above, let x, y be generators of orders 41 and 4 for 41:4, so that the elements of the subgroup are exactly the elements $x^\alpha y^\beta$ for $0 \leq \alpha \leq 40$ and $0 \leq \beta \leq 3$. Let e denote the idempotent

$$e = -\sum_{\alpha,\beta} 2^\beta x^\alpha y^\beta$$

in the group algebra over $GF(5)$—this is actually the block idempotent for one of the faithful irreducible characters (χ_3, say) of the quotient C_4. Then for any FG-module V we obtain a corresponding $eFGe$-module Ve, which is the subspace of e-fixed points. In particular, if V is one of the two 124-dimensional irreducibles, then Ve has dimension 4, while in the other case it has dimension 0.

Now we let V be the mod 5 permutation module on 198400 points, and condense to Ve, which turns out to have dimension 1146—in fact, this dimension can be easily calculated from the character table, since it is the number of copies of χ_3 occurring in the restriction of the permutation character to the subgroup 41:4. Then we find a submodule (actually of dimension 10) of Ve which contains a 4-dimensional constituent. Lifting back to V we find that we generate a submodule of dimension 1271 which turns out to be uniserial with constituents 1023, 124, $\overline{124}$ (the dual of 124) in ascending order. In particular, there is a uniserial, self-dual, quotient module of dimension 248. It is now easy to make all the faithful self-dual 248-dimensional representations of $Sz(32)$ over $GF(5)$. Up to isomorphism there are just two—the uniserial one just constructed, and the direct sum of the two 124-dimensional representations.

7 The subgroup $2^{5+5}{:}31$

Our main task is to find the isomorphism between the subgroups of shape $2^{5+5}{:}31$ in the two groups $Sz(32)$ and $E_8(5)$.

Now we know that there are two such groups in $2^{5+10}L_5(2)$, but only one of these is isomorphic to the Borel subgroup of $Sz(32)$. We first define 'standard generators' for this subgroup as follows. First we choose an element g of order 31, and determine its minimal polynomial $m(x)$ in its action on the normal subgroup of order 2^5. Replacing g by a suitable power of itself, we may assume that $m(x) = x^5 + x^2 + 1$. The minimal polynomial $m'(x)$ of the action of g on the quotient $2^5 = 2^{5+5}/2^5$ is now determined—since this is different in the two subgroups $2^{5+5}{:}31$ in $2^{5+10}{:}31$, this enables us to distinguish the two cases, and to choose the right one. We find that in one case g^3 has minimal polynomial $m(x)$ in its action on the quotient, while in the other case g^9 has this property. The one which is a subgroup of $Sz(32)$ is the former. Moreover, by replacing one element by a suitable power of itself if necessary, we can ensure that the elements of order 31 in the two copies of $2^{5+5}{:}31$ are compatible, in the sense that there is an isomorphism of the groups which carries one element to the other.

Next we choose an element of order 4 to be our second generator. There are exactly 32×31 elements of order 4 in the group, which are fused into 32 orbits of length 31 under conjugation by our first generator (of order 31). Since all elements in one orbit are equivalent, we have 32 essentially distinct possibilities, which we distinguish computationally by calculating a 'fingerprint' (in the sense of Parker [21]) on one of the 124-dimensional representations. We then look at all 32 possibilities to determine the one which is a group isomorphism.

8 Standard basis

Having found the isomorphism between the two groups 2^{5+5}:31 at the abstract level, we now need to find it at the matrix level. There is a standard algorithm for doing this, called the 'standard basis algorithm', described by Parker [21]. The output of this algorithm is a matrix whch conjugates one of these matrix groups to the other. In particular, this provides an explicit verification that the two groups are isomorphic, and hence that $E_8(5)$ contains a subgroup isomorphic to the Borel subgroup of $Sz(32)$.

Note that the representation restricts to this subgroup as the direct sum of two (mutually dual) 124-dimensional submodules. Thus the centralizer of 2^{5+5}:31 in the general linear group is $C_4 \times C_4$, consisting of scalar multiplications on the two constituents. Conjugating by these centralizing elements (modulo scalars) fixes the isomorphism, but makes an orbit of four copies of $Sz(32)$, all of which contain the same subgroup 2^{5+5}:31, in the case when $Sz(32)$ acts indecomposably. On the other hand, in the direct sum case, there is a unique such copy of $Sz(32)$.

Thus we have precisely five cases to check. In each case we have groups $Sz(32)$ and $E_8(5)$ intersecting in at least 2^{5+5}:31, and we need to check whether the whole of the group $Sz(32)$ is contained in the given group $E_8(5)$. It is easy to eliminate four cases, by taking the product of an element of $Sz(32)$ and an element of $E_8(5)$, and finding its order—or, raising it to the power of the exponent of $E_8(5)$, and checking that the result is not the identity matrix. In the remaining case, the result is the identity, which strongly suggests that $Sz(32)$ is contained in $E_8(5)$, but does not prove it.

9 The proof

We now have to check that our extra generator for $Sz(32)$ is contained in $E_8(5)$, by verifying that it preserves the multiplication on the Lie algebra L. For technical reasons it is easier to check the coalgebra structure instead. In representation theoretic terms, we can describe the algebra structure as a homomorphism from $L \otimes L$ to L, in other words as a quotient of $L \otimes L$ which is isomorphic to L. But L is self-dual, and therefore so is $L \otimes L$, which means that $L \otimes L$ contains a submodule isomorphic to L.

More explicitly, we can express elements of $L \otimes L$ as 248×248 matrices, on which G acts by conjugation in the usual way. Moreover, the 248-dimensional submodule N is spanned by the matrices $\mathrm{ad}(v)$ where $v \in L$, or indeed where v runs through a Chevalley basis for L. We merely have to check that N is invariant under our extra generator for $Sz(32)$. But this is an elementary exercise in Gaussian elimination.

More explicitly still, we take the 248 matrices $\mathrm{ad}(e_i)$, where e_i runs through a Chevalley basis for L, and conjugate each by the group element g. (Note that $\mathrm{ad}(e_i g) = g^{-1}\mathrm{ad}(e_i)g$ so this is the correct action.) We write out the 248×248 matrices as vectors of length $248 \times 248 = 61504$, and put the resulting basis for N into echelon form. Now it is easy to check whether $\mathrm{ad}(e_i g)$ is in this 248-space or not.

For our peace of mind, we also checked that the 248-space was invariant under the generators of $E_8(5)$, as well as checking that it was *not* invariant under any of the other four copies of $Sz(32)$ which contain the same 2^{5+5}:31.

10 The outer automorphism

In fact, it is now easy to see that $Sz(32)$:5 is contained in $E_8(5)$. For, the normalizer in $GL_{248}(5)$ of the group 2^{5+5}:31 is just $4 \times 4 \times 2^{5+5}$:31:5, so there is a unique extension of 2^{5+5}:31 to 2^{5+5}:31:5, which is contained in both $Sz(32)$:5 and $E_8(5)$.

We also checked this computationally, by working with $Sz(32)$:5 throughout, rather than $Sz(32)$, and carrying out an explicit check as above.

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