Covering and automorphism groups of $U_6(2)$

Ibrahim A. I. Suleiman
Department of Mathematics and Statistics,
Mu'tah University,
61710 Al-Karak
Jordan
and
Robert A. Wilson
School of Mathematics and Statistics,
The University of Birmingham,
England

published in Quart. J. Math. (Oxford) 48 (1997),511–517

Abstract

We describe the construction of explicit representations of all the covering groups of $U_6(2)$, and then show how to adjoin the outer automorphisms to these groups. The full envelope of the group is $(2^2 \times 3){\cdot}U_6(2){:}S_3$, which can be represented in 720 dimensions over the field of order 7.

1 Introduction

A number of groups of Lie type have so-called 'exceptional covers', that is, there is a non-trivial p-part to their Schur multiplier, where p is the defining characteristic of the group. One particularly interesting case is the unitary group $U_6(2)$, which has, in addition to its 'generic' multiplier of 3 and outer automorphism group S_3, a further 2^2 of multiplier. This means that the full envelope of the group has shape $(2^2 \times 3){\cdot}U_6(2){:}S_3$, and cannot be constructed by purely classical means. Our aim in this paper is to show how any desired part of this group may be constructed, starting from the natural representation of $SU_6(2) \cong 3{\cdot}U_6(2)$, and a representation of $2{\cdot}U_6(2)$ which can be obtained as a subgroup of the sporadic Fischer group Fi_{22}.

We exploit the idea of 'standard generators' [10] in two ways. First, given a pair (a, b) of standard generators for a group G, we can find images (a', b') of (a, b) under outer automorphisms, and hence construct representations of $\mathrm{Aut}(G)$. Second, such an automorphism of G may be lifted to a 'near-automorphism' of a covering group, in order to construct other covers. In this case, from the double cover $2{\cdot}U_6(2)$, we can construct the cover $2^2{\cdot}U_6(2)$.

Our notation follows the Atlas of Finite Groups [2]. The matrix calculations were performed using a version of Parker's Meat-axe [4] implemented by M. Ringe [6]. A few character calculations were carried out using GAP [8]. All the representations which we construct here are available on the world-wide-web under the URL http://www.mat.bham.ac.uk/atlas/ [11].

2 Standard generators and automorphisms

Our first task is to study the simple group and its automorphisms. We first define standard generators for $U_6(2)$ to be (a, b) where $a \in 2A$ (that is, a is an element of order 2 in the conjugacy class labelled $2A$ in the Atlas [2]—in classical terms, this is the class of unitary transvections), b has order 7, ab has order 11, and abb has order 18. This defines the pair (a, b) up to automorphisms, and therefore there are exactly six conjugacy classes of such pairs, permuted regularly by the outer automorphism group S_3.

By random search in a suitable representation (constructed below), we found a number of pairs of generators automorphic to (a, b), and by adjoining the corresponding automorphisms to $U_6(2)$, constructed $U_6(2){:}S_3$.

Specifically, we found that $(a', b') = ((abb)^{-4}a(abb)^4, (abababbab)^{-1}bababab bab)$ is the image of (a, b) under an outer automorphism of order 3, while $(a'', b'') = (a, b^{-1})$ is the image under an outer automorphism of order 2.

To construct explicit representations of these groups, we started with the natural representation of $SU_6(2) \cong 3{\cdot}U_6(2)$ of dimension 6 over $GF(4)$. By writing down some random generators, and then conducting a random search, we found that the two matrices in Table 1 give standard generators for $3{\cdot}U_6(2)$, in the sense that they map to standard generators of $U_6(2)$ under the natural quotient map, and also have orders 2 and 7 respectively, so that

Table 1: Standard generators for $SU_6(2)$

$$A = \begin{pmatrix} 0 & 0 & \omega & 1 & \overline{\omega} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \overline{\omega} & 0 & 0 & \overline{\omega} & \omega & 0 \\ 1 & 0 & \omega & 0 & \overline{\omega} & 0 \\ \omega & 0 & \overline{\omega} & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \overline{\omega} & 0 & \overline{\omega} & \overline{\omega} & \overline{\omega} & \omega \\ 1 & \overline{\omega} & 0 & \overline{\omega} & \overline{\omega} & \overline{\omega} \\ 0 & \overline{\omega} & 1 & 0 & 0 & \overline{\omega} \\ 1 & \overline{\omega} & 1 & \omega & \omega & 0 \\ 0 & 1 & \overline{\omega} & 1 & \overline{\omega} & \overline{\omega} \\ 0 & \omega & 1 & 1 & \omega & 1 \end{pmatrix}$$

they are uniquely defined up to automorphisms. These matrices are written with respect to an orthonormal basis.

The exterior cube of this representation is a 20-dimensional irreducible representation of the simple group $U_6(2)$, which can be written over $GF(2)$. Moreover, this representation extends to the full automorphism group $U_6(2){:}S_3$. We use the Meat-axe to write this representation with respect to a suitable 'standard basis', as described in [4]. It is an undocumented feature of this algorithm that if the element f_i of the group algebra (whose null-vector $n(f_i)$ is used to seed the standard-basis program) has coefficients in the prime subfield F_0 of the underlying field, then the new matrices for the representation will be written over the smallest possible field. Thus when we use this method to put the given representation into standard form, it is automatically written over $GF(2)$.

Next we apply the above words to obtain matrices which are conjugate to the original generators under an outer automorphism of order 3. Applying the standard basis algorithm again, we obtain a matrix which conjugates the new generators to the old ones. Adjoining this matrix therefore gives a group which, modulo scalars, is isomorphic to $U_6(2){:}3$. Since we are working over $GF(2)$, there are no non-trivial scalars, and the group is actually isomorphic to $U_6(2){:}3$.

A similar method was used to adjoin an outer automorphism of order 2 to obtain $U_6(2){:}2$. Adjoining both automorphisms at once gives a 20-dimensional representation of $U_6(2){:}S_3$ over $GF(2)$.

Once we have generators for these various automorphism groups, it is fairly easy to make choices of standard generators and check that they are well-defined. Here we use GAP [8] to calculate structure constants from the character tables. If necessary, we then use the Meat-axe to find representatives of all classes of pairs (x, y) of elements with x, y, and xy in the specified conjugacy classes, to determine what other information is required to define (x, y) up to conjugacy.

We make the following definitions, again using Atlas notation for conjugacy classes.

Standard generators for $U_6(2){:}2$ are (c, d) with $c \in 2D$, $d \in 6J$ and cd of order 11.

Standard generators for $U_6(2){:}3$ are (e, f) with $e \in 3D$, f of order 11, ef of order 21, and eff of order 18.

Standard generators for $U_6(2){:}S_3$ are (g, h) with $g \in 2D$, $h \in 6J$ and gh of order 21.

For completeness we note that the outer automorphism of $U_6(2){:}3$ may be realised by mapping (e, f) to (e^{-1}, f^{-1}).

3 The triple cover of $U_6(2)$

We start with the standard generators of $3{}^{\cdot}U_6(2)$ given above, and change to a standard basis in the usual way. We then apply the same words as above to obtain new generators conjugate to the old ones under outer automorphisms. It is not immediately obvious that we can use the same words as in $U_6(2)$, since they are only guaranteed to work modulo the centre. However, the actual words we used give conjugates of the old generators, so the new generators also have orders 2 and 7. Therefore they are standard generators, that is, automorphic to the old generators.

Using first the automorphism of order 3, we easily construct the 6-dimensional representation of $3{}^{\cdot}U_6(2){:}3 \cong GU_6(2)$.

To adjoin the automorphism of order 2 to $3{}^{\cdot}U_6(2)$, we need to take the direct sum of this representation and its dual. Again we write this larger representation with respect to a standard basis, and apply the words to obtain new generators. The usual method produces a representation of a group isoclinic to the one we want. This time we found it necessary to cube the extra generator in order to eliminate the unwanted scalars. Finally, the standard basis method can be used again to write this representation over $GF(2)$.

A similar method can be used to extend $3{}^{\cdot}U_6(2){:}3$ to $3{}^{\cdot}U_6(2){:}S_3$. In this case the automorphism can be chosen to have a particularly nice form. If we define standard generators to be pre-images (E, F) of (e, f) such that F has order 11, then the map $(E, F) \mapsto (E^{-1}, F^{-1})$ induces such an automorphism.

4 The double cover of $U_6(2)$

A faithful 56-dimensional representation of $2{\cdot}U_6(2)$ over $GF(3)$ can be obtained from the 77-dimensional representation of Fi_{22}:2 found in the library [11], since $2{\cdot}U_6(2)$ is the centralizer of an involution in Fi_{22}.

We can use this to construct a representation of $2^2{\cdot}U_6(2)$:3, as follows. We take pre-images A and B in $2{\cdot}U_6(2)$ of standard generators a and b of $U_6(2)$. In order to specify A and B up to automorphisms, we demand that B has order 7 and AB has order 11. Now we apply the same words that give an outer automorphism of order 3 of $U_6(2)$. This automorphism cycles the three double covers of $U_6(2)$, and can therefore be used to obtain three 56-dimensional representations of $2^2{\cdot}U_6(2)$, each with a different involution in the kernel. By taking the direct sum of these three representations, we obtain a faithful representation of $2^2{\cdot}U_6(2)$, which is invariant under the outer automorphism of order 3.

We can now adjoin this outer automorphism, in the same way as before. Indeed, the representation is also invariant under the outer automorphism of order 2, so we can adjoin both at once, and obtain a representation of $2^2{\cdot}U_6(2)$:S_3. The only thing we need to worry about is that we are now working over $GF(3)$, and hence we may have an extra scalar matrix of order 2. In fact, we found that this problem did not arise.

5 Permutation representations

The representations we have constructed so far are in characteristic 2 or 3, where it is impossible to construct a faithful irreducible representation of the sixfold cover, for example. Indeed, the obvious place to start looking for such a representation is over a field with scalars of order 6, such as $GF(7)$.

First note that the largest subgroup $U_5(2)$ of $U_6(2)$ has trivial multiplier, so any representation induced up from $U_5(2)$ to a covering group of $U_6(2)$ will be faithful. In particular, there exist faithful permutation representations of $2{\cdot}U_6(2)$ on 1344 points, and of $3{\cdot}U_6(2)$ on 2016 points. Moreover, both are easy to make, as the subgroup $U_5(2)$ fixes a vector both in the natural 6-dimensional representation of $3{}^{\cdot}U_6(2)$, and in the 56-dimensional representation of $2{}^{\cdot}U_6(2)$ used above.

If we chop up these permutation representations over $GF(7)$, then we obtain in particular the 56-dimensional irreducible for $2{}^{\cdot}U_6(2)$. However, the smallest representation for the triple cover, of dimension 21, does not appear. Thus we decided to construct this representation from scratch, using the approach described in [5]. As this method is by now fairly standard, we do not give full details.

6 The 21-dimensional representation of $3{}^{\cdot}U_6(2)$ over $GF(7)$

Our plan was to start with the representation restricted to $3 \times U_5(2)$, and then restrict further to a subgroup $3 \times 3 \times U_4(2)$. This then extends to $3 \times S_3 \times U_4(2)$ inside $3{}^{\cdot}U_6(2)$, and both these subgroups are maximal, so together they generate the group.

First we need to use the character tables in the Atlas [2] to see how the representation restricts to these various subgroups. This is possible as their orders are all coprime to 7, so the 7-modular characters are the same as the ordinary characters. As in the Atlas, we label characters by their degrees followed by a distinguishing letter, and use the same notation for the corresponding representations and modules.

From the character tables we see that the restriction to $U_5(2)$ is $10a+11a$, with the centre (of order 3) of $3{}^{\cdot}U_6(2)$ acting as scalars. Restricting to $U_4(2)$, the 10-dimensional representation breaks up as $5a+5b$, and the representation $11a$ breaks up as $5b + 6a$. This subgroup $U_4(2)$ centralizes an elementary abelian group of shape 3^2 in $3{}^{\cdot}U_6(2)$, and the four irreducible constituents are visibly the eigenspaces of the central 3^2. Clifford's theorem implies that these eigenspaces are permuted by the involution which extends the group to $3 \times S_3 \times U_4(2)$. Since this involution is in the conjugacy class $2A$ in $3{}^{\cdot}U_6(2)$, it has character value -11, and therefore it must negate the spaces $5a$ and $6a$, and interchange the two $5b$-spaces. Without loss of generality, it bodily interchanges them without any scale factor.

Turning now from the characters to the problem of constructing the corresponding representations, our next task is to make the representation of $U_5(2)$. It turns out that the 11-dimensional constituent can be obtained in any desired characteristic as follows. First take the 24-dimensional representation of Conway's group $2{\cdot}Co_1$ given by its action on the Leech lattice. This has been written with respect to an integral basis by Richard Parker. This group contains a maximal subgroup $6{\cdot}Suz{:}2$, which in turn contains a subgroup $U_5(2)$. In characteristic 0 the given representation restricts to $U_5(2)$ as $1a+1a+11a+11b$, and by reducing modulo 7 we obtain our desired representation.

The 10-dimensional constituent corresponds to a 5-dimensional quaternionic representation, generators for which are given on p. 72 of the Atlas [2]. By identifying the quaternionic group $2{\cdot}A_4$ with a group of 2×2 matrices over $GF(7)$, we can write down 10×10 matrices over $GF(7)$ which generate $U_5(2)$.

Then we make the direct sum of these two representations, and find a subgroup $U_4(2)$ inside $U_5(2)$. The action of the commuting 3^2 is clear and can be written down. As noted above, the irreducible constituents for $U_4(2)$ are exactly the eigenspaces for 3^2. Furthermore, the action of the additional involution on these eigenspaces is also clear. All we need to do is change basis so that the four subspaces are visible, and the two copies of $5b$ are written with respect to the same basis. Then the extra involution can be simply written down, as

$$\begin{pmatrix} -I_5 & 0 & 0 & 0 \\ 0 & 0 & I_5 & 0 \\ 0 & I_5 & 0 & 0 \\ 0 & 0 & 0 & -I_6 \end{pmatrix}.$$

The usual random search then produces standard generators for the group in this representation.

It is worth noting that this procedure will work in any characteristic not dividing the order of $U_5(2)$, including characteristic 0.

7 The full cover

It is now possible to make a faithful irreducible representation of $6{\cdot}U_6(2)$ by taking the tensor product of the 21-dimensional representation of $3{\cdot}U_6(2)$ with the 56-dimensional representation of $2{\cdot}U_6(2)$. This can then be chopped up with the Meat-axe, and the smallest faithful irreducible, of dimension 120, extracted from it.

We can then carry out essentially the same procedures as already described for the double and triple covers, to make the full cover $(2^2 \times 3){\cdot}U_6(2)$, and to adjoin the automorphisms to it. In particular, we can make a 720-dimensional representation of the full envelope $(2^2 \times 3){\cdot}U_6(2){:}S_3$.

Acknowledgements

We would like to thank the London Mathematical Society and the School of Mathematics and Statistics in the University of Birmingham, for partial support of a visit to Birmingham by the first author, during which this work was done. We also thank the SERC (now EPSRC) for a grant to purchase a computer on which most of the calculations were performed.

References

[1] R. W. Carter, *Simple groups of Lie type*, Wiley, 1972.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[3] C. Jansen, K. Lux, R. A. Parker and R. A. Wilson, *An Atlas of Brauer Characters*, Clarendon Press, Oxford, 1995.

[4] R. A. Parker, The computer calculation of modular characters (The 'Meat-axe'), *Computational Group Theory (ed. M. D. Atkinson)*, pp. 267–274. Academic Press, 1984.

[5] R. A. Parker and R. A. Wilson, Computer construction of matrix representations of finite groups over finite fields, *J. Symbolic Comput.* **9** (1990), 583–590.

[6] M. Ringe, *The C Meat-axe 2.3, documentation*, RWTH Aachen, 1995.

[7] A. J. E. Ryba, Computer condensation of modular representations, *J. Symbolic Comput.* **9** (1990), 591–600.

[8] M. Schönert et al., *GAP 3.4 Manual (Groups, Algorithms, and Programming)*, RWTH Aachen, 1994.

[9] I. A. I. Suleiman, Modular representations of some finite simple groups, Ph. D. Thesis, University of Birmingham, 1990.

[10] R. A. Wilson, Standard generators for sporadic simple groups, *J. Algebra*, to appear.

[11] R. A. Wilson et al., A world-wide-web atlas of finite group representations, http://www.mat.bham.ac.uk/atlas.