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A QUATERNIONIC CONSTRUCTION OF E_7

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Abstract. We give an explicit construction of the simply-connected compact real form of the Lie group of type E_7, as a group of 28×28 matrices over quaternions, acting on a 28-dimensional left quaternion vector space. This leads to a description of the simply-connected split real form, acting on a 56-dimensional real vector space, and thence to the finite quasi-simple groups of type E_7. The sign problems usually associated with constructing exceptional Lie groups are almost entirely absent from this approach.

1. Introduction

The simple Lie groups over $\mathbb{C}$ are of eight types: three classical (orthogonal, unitary and symplectic), and five exceptional (G_2, F_4, E_6, E_7 and E_8). Over $\mathbb{R}$, each type is divided into a number of 'real forms': for example, the real forms of the orthogonal groups are parametrized by the signature of the underlying quadratic form (up to sign). In every case, there is exactly one compact real form, and one split real form, and there may be others in between. In the case of the orthogonal groups, the compact real form has a positive-definite quadratic form, while the split real form has quadratic form with the numbers of positive and negative terms being as nearly equal as possible.

Thus the compact real form of the orthogonal group $\mathrm{O}(n)$ acts on a real n-space preserving the quadratic form $\sum_{r=1}^{n} x_r^2$. Similarly the compact real form of the unitary group $\mathrm{SU}(n)$ acts on a complex n-space preserving the Hermitian form $\sum_{r=1}^{n} x_r\overline{x_r}$. And the compact real form of the symplectic group $\mathrm{Sp}(n)$ acts on a quaternionic n-space preserving the quaternionic norm $\sum_{r=1}^{n} x_r\overline{x_r}$.

Each real form may further divide into different isomorphism types: there is always an adjoint group (acting on the Lie algebra), and a simply-connected group (which may or may not be the same), and sometimes others in between. In the case of the orthogonal groups, the adjoint group is the projective group $\mathrm{PO}(n)$, given by the action by conjugation on a suitable space of $n \times n$ matrices, and the simply-connected group is the spin group, that is a double cover of the orthogonal group, acting on the Clifford algebra.

The exceptional groups are generally constructed in the adjoint action of the split real form, as this permits the most uniform approach. However, this has several drawbacks. First, the adjoint representation is not the smallest (except in the case

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of E_8); second, the centre of the group acts trivially; and third, the compact real form often has nicer properties. The following table explains the situation:

Type	G_2	F_4	E_6	E_7	E_8
Lie algebra dimension	14	52	78	133	248
Smallest dimension	7	26	27	56	248
Centre	1	1	3	2	1

There are a number of constructions of the minimal representations in the literature, sometimes over $\mathbb{C}$ or $\mathbb{R}$, sometimes over finite fields, with or without a restriction on the characteristic. For example, in 1901 L. E. Dickson constructed the finite groups of type E_6 by defining an invariant cubic form with 45 terms in 27 variables [5, 6]. The same construction works over arbitrary fields. He later constructed G_2 as the automorphism group of the Cayley numbers (octonions) [4]. The interpretation of F_4 as the automorphism group of the 27-dimensional exceptional Jordan algebra came later (see for example [7] for an exposition). This algebra consists of 3×3 Hermitian matrices over octonions, with product $AB + BA$, and leads to an interpretation of Dickson's cubic form as the determinant of such matrices (in the case when the split real form of the octonions is used).

The 56-dimensional representation of E_7 was constructed by Brown [2], and further studied by Aschbacher [1] and by Cooperstein [3], who were principally interested in the finite case. Thus they did not make use of the fact that the representation is symplectic, that is, writable over quaternions in half the number of dimensions. In this paper we significantly simplify the treatment of E_7 in [2, 1, 3] by exploiting the quaternionic structure to the full. We first construct the compact real form, and only later convert to the split real form in order to reduce modulo p. This approach brings out the rather striking fact that this makes (the simply-connected split real forms of) F_4, E_6 and E_7 respectively 26-dimensional real, 27-dimensional complex, and 28-dimensional quaternionic. In particular, the exceptional Jordan algebra is not the end of the line, but is only part of a much richer 28-dimensional structure.

In Section 2 we define a certain group G to be the group generated by certain explicit 28×28 matrices over quaternions. In Section 3 we prove that G is the simply-connected compact real form of E_7. In Section 4 we discuss the invariant quadrilinear form, and in Section 5 we describe some subgroups. Finally we consider the split real form and the finite groups of type E_7 in Section 6.

2. The action of the root groups

First we describe the labelling of the 28 quaternionic coordinates in terms of the 28 pairs of opposite minimal vectors in the dual E_7 lattice E_7^*. We label the 7 coordinates of $\mathbb{R}^7$ by the elements $0, 1, \ldots, 6$ of the field $\mathbb{F}_7$, and may then take the 126 roots of the E_7 lattice to be the images under sign changes and cyclic permutations of the coordinates of the following vectors:

- 14 images of $(2, 0, 0, 0, 0, 0, 0)$;
- 112 images of $(1, 0, 0, 1, 0, 1, 1)$.

Then the 56 minimal vectors of E_7^* (multiplied by 2 for convenience) are the images under sign-changes and rotations of $(0, 1, 1, 0, 1, 0, 0)$.

The Coxeter group of type E_7 is generated by the 63 reflections in the roots. For example, reflection in $\pm(2, 0, 0, 0, 0, 0, 0)$ negates coordinate 0, while reflection in $(1,0,0,1,0,1,1)$ fixes coordinates $1,2,4$ and acts as the matrix

$$-\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

on coordinates $0,3,5,6$. The monomial subgroup of the Coxeter group is of shape 2^7:$\mathrm{PSL}_3(2)$, and is generated by the sign-changes together with the elements which permute the coordinates by $t \mapsto t+1$, and $t \mapsto 2t$ and $(1,2)(3,6)$.

Now label the 28 pairs of minimal vectors in E_7^* by

$$\begin{array}{rcl} h_0 & = & \{\pm(0,1,1,0,1,0,0)\}, \\ i_0 & = & \{\pm(0,1,-1,0,-1,0,0)\}, \\ j_0 & = & \{\pm(0,-1,1,0,-1,0,0)\}, \\ k_0 & = & \{\pm(0,-1,-1,0,1,0,0)\}, \end{array}$$

where adding 1 to the subscript (modulo 7) corresponds to rotating the coordinates backwards. We use the same labels for the 28 coordinate vectors in the left vector space $\mathbb{H}^{28}$, and use corresponding capital letters for the coordinates of a typical vector, thus

$$(H_0, I_0, J_0, K_0, H_1, \ldots, J_6, K_6) = H_0h_0 + I_0i_0 + \cdots + K_6k_6 \in \mathbb{H}^{28}.$$

We are now ready to describe the action of 63 copies of $\mathrm{SU}(2)$ on this space, one for each of the 63 pairs of opposite roots. First we take the roots $\pm(2,0,0,0,0,0,0)$. This copy of $\mathrm{SU}(2)$ fixes all the coordinates with a subscript $0,3,5,6$. Now for any element $q \in \mathrm{SU}(2) \subset \mathbb{H}$, that is $q = z + wj$ with $z, w \in \mathbb{C} = \mathbb{R}[i]$ and $q\overline{q} = z\overline{z} + w\overline{w} = 1$, we define an action of q as right-multiplication by

$$\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$$

on each of the quaternionic 2-vectors (H_1, I_1), (H_2, J_2), and (H_4, K_4), and as

$$\begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & \overline{z} \end{pmatrix}$$

on each of (J_1, K_1), (K_2, I_2) and (I_4, J_4). To prove that this indeed defines an action it suffices to check that the matrix product

$$\begin{pmatrix} u & vj \\ vj & u \end{pmatrix}\begin{pmatrix} z & wj \\ wj & z \end{pmatrix} = \begin{pmatrix} uz - v\overline{w} & (uw + v\overline{z})j \\ (uw + v\overline{z})j & uz - v\overline{w} \end{pmatrix}$$

corresponds to the quaternion product

$$(u + vj).(z + wj) = (uz - v\overline{w}) + (uw + v\overline{z})j,$$

since the other action is obtained from this by conjugation by j.

We may apply a cyclic permutation of the 7 subscripts $t \in \mathbb{F}_7$ (i.e. map $t \mapsto t+1$), to get a total of seven such fundamental $\mathrm{SU}(2)$s. It is easy to check that these $\mathrm{SU}(2)$s commute with each other, since the matrices

$$\begin{pmatrix} u & vj & & \\ vj & u & & \\ & & \overline{u} & \overline{v}j \\ & & \overline{v}j & \overline{u} \end{pmatrix}, \begin{pmatrix} z & & wj & \\ & \overline{z} & & \overline{w}j \\ wj & & z & \\ & \overline{w}j & & \overline{z} \end{pmatrix}$$

commute. Notice that there is also a symmetry of order 3 permuting these seven copies of SU(2), acting as $t \mapsto 2t$ on the suffices, and as (I, J, K) on the letters. This symmetry will be called 'triality', for reasons which will become clear later.

Next consider the element R_0 corresponding to $q = j$ in the SU(2) just constructed. It maps

$$\begin{array}{rcl} (H_1, I_1, J_1, K_1) & \mapsto & (I_1 j, H_1 j, K_1 j, J_1 j), \\ (H_2, I_2, J_2.K_2) & \mapsto & (J_2 j, K_2 j, H_2 j, I_2 j), \\ (H_4, I_4, J_4, K_4) & \mapsto & (K_4 j, J_4 j, I_4 j, H_4 j). \end{array}$$

The corresponding action in the Coxeter group is to negate coordinate 0 in the vectors $(1, 1, 0, 1, 0, 0, 0)$, $(1, 0, 1, 0, 0, 0, 1)$ and $(1, 0, 0, 0, 1, 1, 0)$. Hence by using these sign-changes and rotations of the seven coordinates we have transitivity on the remaining 56 pairs of roots. It is enough therefore to specify the action of one more copy of SU(2), for example the one which acts as right-multiplication by

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}$$

on each of the quaternionic 2-vectors (H_1, K_3), (H_2, I_6), and (H_4, J_5), and as

$$\begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$$

on each of (H_3, J_1), (H_6, K_2) and (H_5, I_4). Notice that this copy of SU(2) is centralized by the triality element.

Now if $R_1, \ldots, R_6$ denote the images of R_0 under repeatedly subtracting 1 from the subscripts modulo 7, then we can easily check that this copy of SU(2) is centralized by R_1, R_2, and R_4, and normalized by $R_0R_3R_5R_6$. Hence there are exactly 56 images under conjugation by the R_t and the rotation.

(The calculations are as follows. First, R_1 maps (H_1, K_3) to $(J_1 j, H_3 j)$ and (H_3, J_1) to $(K_3 j, H_1 j)$ while centralizing all the other 2-spaces on which the SU(2) acts. Since

$$\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix} = \begin{pmatrix} z & -w \\ \overline{w} & \overline{z} \end{pmatrix}$$

we see that R_1 centralizes this SU(2). The symmetry $(1, 2, 4)(3, 6, 5)(I, J, K)$ shows the same is true for R_2 and R_4. Similarly, $R_0R_6R_5R_3$ maps the pairs

$$(H_1, K_3), (H_2, I_6), (H_4, J_5)$$

to the negatives of

$$(J_1, H_3), (K_2, H_6), (I_4, H_5)$$

and vice versa, and we calculate

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\overline{z} & \overline{w} \\ -w & -z \end{pmatrix}.$$

Hence $R_0R_6R_5R_3$ normalizes the SU(2), by mapping $z + wj \mapsto -\overline{z} + \overline{w}j$.)

We have thus defined explicitly the actions of 63 root groups SU(2) on $\mathbb{H}^{28}$. Let G be the group generated by these 63 copies of SU(2). It remains to prove that G is a Lie group of type E_7, rather than the whole of Sp(28), or something else entirely.

3. The torus and the Weyl group

Each of the 63 root SU(2) groups has a torus (given by $q = z \in \mathbb{C}$) and a Weyl group (given by $q = j$ modulo the torus), and we can put all these together to get the torus and Weyl group for G. The two types of tori are exemplified by

- multiplying by z on $H_1, H_2, H_4, I_1, J_2, K_4$ and by $\overline{z}$ on $J_1, K_2, I_4, K_1, I_2, J_4$;
- multiplying by z on $H_1, H_2, H_4, H_3, H_6, H_5$ and by $\overline{z}$ on $K_3, I_6, J_5, J_1, K_2, I_4$.

It is straightforward to check that these 63 tori together generate a 7-dimensional torus. Indeed, the seven tori of the first type are independent (since they lie in seven commuting SU(2) subgroups), and each of the other elements listed is the square root of a suitable product of four of the first type. But $\mathbb{C}$ is algebraically closed, so all these elements are contained in the 7-dimensional torus.

The two types of reflection act as follows on the quaternionic coordinates, where the symbol (X, Y) denotes the map $X \mapsto Y \mapsto -X \mapsto -Y \mapsto X$.

- $(H_1, I_1 j)(H_2, J_2 j)(H_4, K_4 j)(J_1, K_1 j)(K_2, I_2 j)(I_4, J_4 j)$;
- $(K_3, H_1)(I_6, H_2)(J_5, H_4)(H_3, J_1)(H_6, K_2)(H_5, I_4)$.

Now the permutation action on the quaternionic coordinates of the Weyl group in each SU(2) is clearly the same as the action on the 28 pairs of minimal vectors in E_7^* of the corresponding reflection in the Coxeter group. Hence we see that the Weyl group is indeed isomorphic to the Coxeter group of type E_7, that is, to $2 \times \mathrm{Sp}_6(2)$. The central involution multiplies every coordinate by j.

It turns out that the part of the Weyl group which preserves the decomposition of the 28-dimensional quaternionic space as a sum of seven 4-spaces is a subgroup of shape $2^7\mathrm{PSL}_3(2) = 2.2^3.2^3.\mathrm{PSL}_3(2)$. This group is generated (modulo the torus) by the following elements, and since it is a maximal subgroup of the Weyl group, it is the stabilizer of the decomposition, as claimed.

(1) right-multiplication by j on all coordinates;
(2) negating coordinates labelled 0, and permuting coordinates as $(H, I)(J, K)$ when the label is 4 or 6, as $(H, J)(K, I)$ when the label is 1 or 5, and as $(H, K)(I, J)$ when the label is 2 or 3.
(3) acting as $j(H, I)(J, K)$ on coordinates labelled 1, as $j(H, J)(K, I)$ when the label is 2, and as $j(H, K)(I, J)$ when the label is 4; thus $H_1 \mapsto I_1 j \mapsto -H_1$ and so on;
(4) cyclically permuting the labels by $t \mapsto t + 1$ (where $t \in \mathbb{F}_7$);
(5) permuting the labels by $t \mapsto 2t$ together with (I, J, K);
(6) $(H_0, -H_0)(I_0, -J_0)(K_0, -K_0)(I_5, K_5)$
$(H_1, H_2)(I_1, J_2)(J_1, K_2)(K_1, I_2)$
$(H_3, H_6)(I_3, K_6)(J_3, J_6)(K_3, I_6)$

As some of these elements (particularly (3) and (6)) are slightly awkward to apply in practice, for example in getting the correct power of j in every coordinate, we shall often restrict to a smaller symmetry group $2 \times 2^3{:}7{:}3$, generated by the elements (1), (2), (4), and (5).

Our construction shows that this Weyl group is generated by any one reflection of each type together with the element (4) of order 7. We proved in the previous section that the first reflection, that is the element (3), preserves the set of 63 root SU(2)s.

Next we show that the second reflection also preserves the set of 63 root groups. We already showed that this reflection commutes with R_1, R_2, R_4 and $R_0R_6R_5R_3$, as well as the triality element (5). Now it is easy to see that this centralizing group of order 48 has orbits of sizes $(1+3+3)+(1+1+3+3+24+24)$ on the $7+56$ root groups. Hence it is enough to check the action of our reflection on one group in each orbit, which is now an easy calculation. For example, this reflection maps the coordinates

$$(H_2, I_2; H_3, J_3; H_5, K_5), (J_2, K_2; K_3, I_3; I_5, J_5)$$

respectively to

$$(-I_6, I_2; J_1, J_3; I_4, K_5), (J_2, -H_6; H_1, I_3; I_5, H_4),$$

which goes under a suitable rotation of element (3) to

$$(-I_6, H_2j; J_1, H_3j; I_4, H_5j), (K_2j, -H_6; H_1, K_3j; J_5j, H_4).$$

The action of the root group on the various 2-spaces can be computed by conjugation of matrices: for example the action on (H_1, K_3) is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & \overline{z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -j \end{pmatrix} = \begin{pmatrix} \overline{z} & \overline{w} \\ -w & z \end{pmatrix}.$$

Similar conjugations on the other five 2-spaces, and re-writing the quaternion $z+wj$ as $\overline{z'} + \overline{w'}j$, gives us the second copy of SU(2) defined above. (The rest of this calculation is left as an exercise for the reader. The reflection swaps two of the orbits of size 1, and two pairs of orbits of size 3, while centralizing the rest, so there are six cases to check, of which we have sketched one.)

It follows from these calculations that G is generated by a single SU(2) together with the Weyl group. In particular, it is of type E_7. Clearly it must be the simply-connected compact real form.

4. The quadrilinear form

The cited references [2, 3, 1] all define E_7 (in the relevant context: usually over a finite field) as the stabilizer of a pair of forms, one bilinear, the other quadrilinear in the 56 complex coordinates. Although we do not need this as part of our definition, it is useful for further investigations to have the (totally symmetric) quadrilinear form explicitly. We have already taken care of the bilinear form by ensuring that our representation commutes with left-multiplication by j, so is symplectic. In fact, the quadrilinear form is invariant not only under the compact real form, but under the whole of the complexification $E_7(\mathbb{C})$.

To describe this complexification, we take as our basis for the complex 56-space the original quaternionic basis $\{h_t, i_t, j_t, k_t\}$ together with the multiples by j, that is $\{jh_t, ji_t, jj_t, jk_t\}$. The corresponding complex coordinates of a vector will be written X'_t, X''_t, where $X_t = X'_t + X''_t j$. The symbol j now loses its quaternionic meaning, and just acts as a formal symbol permuting coordinates (up to sign), though still with the understanding that $j^2 = -1$. With this interpretation, the Weyl group permutes the complex coordinates, up to sign, so it suffices to deal with one root group. We re-compute the action of $\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$ on (H_1, I_1), etc., to be $\begin{pmatrix} z & \overline{w} \\ -w & \overline{z} \end{pmatrix}$ on both (H'_1, I''_1) and (I'_1, H''_1). Similarly, the action of $\begin{pmatrix} \overline{z} & \overline{w}j \\ \overline{w}j & z \end{pmatrix}$ gives the transpose-inverse matrix $\begin{pmatrix} \overline{z} & w \\ -\overline{w} & z \end{pmatrix}$ acting on (J'_1, K''_1) etc.. Next we extend from SU(2) to SL$(2,\mathbb{C})$, by allowing all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1 in this place (and, of course, the transpose-inverse $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ in the appropriate places). Note that conjugation by j can now be interpreted as the transpose-inverse map.

It turns out that we may then define the quadrilinear form by defining it on three quadruples of complex basis vectors:

$$\begin{array}{rcl} [h_0, h_0, jh_0, jh_0] &=& -1 \\ [h_0, jh_0, i_0, ji_0] &=& 1/2 \\ [h_0, i_0, j_0, k_0] &=& 1 \end{array}$$

and images under the action of the Weyl group. Equivalently, the quartic form is defined as the sum of monomials of the form

$$-(H_0'H_0'')^2, \frac{1}{2}H_0'H_0''I_0'I_0'', H_0'I_0'J_0'K_0'.$$

Since the Weyl group can be taken to permute our 56 coordinates, up to signs, it permutes exactly 28 monomials which are images of the first of those listed, since any sign-changes cancel out. Similarly, the second gives rise to 378 terms, one for each pair of the 28 coordinates. In order for our quartic form to be well-defined, we must show that $H_0'H_0''I_0'I_0''$ is not negated by any element in the part of the Weyl group which fixes the pair $\{h_0, i_0\}$ of quaternionic coordinates. This is a straightforward calculation. Indeed, the relevant subgroup of the Weyl group is 2×2^5S_5. This subgroup has a centre of order 4, in which the central involution of the whole Weyl group acts as right-multiplication by j, and the central reflection swaps h_0 with i_0. The outer half of S_5 may be taken to act trivially.

The third type of monomial is more interesting. The 630 monomials of this type fall into three orbits, of sizes $14+168+448$, under the action of the subgroup $2^7{:}7{:}3$ described above. These are represented respectively by $H_0'I_0'J_0'K_0'$, $H_0'I_0'I_1'K_1'$ and $H_0'I_3'J_6'K_5'$. For convenience we also give the orbits under $2 \times 2^3{:}7{:}3$:

- 14 images of $H_0'I_0'J_0'K_0'$;
- 42 images of each of $H_0'I_0'I_1'K_1'$, $H_0'I_0'H_1''J_1''$, $J_0'K_0'H_1'J_1'$ and $J_0'K_0'I_1''K_1''$;
- 112 images of $H_0'I_3'J_6'K_5'$;
- 336 images of $H_0'I_3'H_6''J_5''$.

(Notice that, although the coefficients of all the displayed monomials are $+1$, nevertheless our symmetry group has minus signs, which introduce many minus signs into the quartic form. There are 105 terms with all single dashes, 105 terms with all double dashes, and 420 terms with two of each.)

In this third case, the relevant subgroup of the Weyl group has shape $2\times[2^5]S_3S_4$, and its action on the 4-dimensional quaternionic space $\langle h_0, i_0, j_0, k_0\rangle$ is $2 \times S_4$, generated by right-multiplication by j together with all coordinate permutations. This completes the proof that the quartic form (and the corresponding quadrilinear form) is well-defined, and (therefore, by definition) invariant under the action of the Weyl group.

Now to prove that this form is invariant under the complexification of G, we must show it is invariant under an arbitrary element of our fundamental SL$(2,\mathbb{C})$, corresponding to a pair of opposite roots, say $(\pm(2,0,0,0,0,0)$. First we compute the orbits of the root stabilizer in the Weyl group, on the quadruples used in the definition of the form. There are just two orbits on the 28 images of the first quadruple, corresponding to the quaternionic 1-spaces spanned by h_0 and h_1. The first case is obviously fixed, while the second gives, as required,

$$\begin{array}{rcl}[h_1,h_1,jh_1,jh_1] & = & [ah_1+bji_1,ah_1+bji_1,djh_1+ci_1,djh_1+ci_1]\\ & = & (ad)^2[h_1,h_1,jh_1,jh_1]+(bc)^2[ji_1,ji_1,i_1,i_1]\\ & & +abcd([h_1,ji_1,jh_1,i_1]+[h_1,ji_1,i_1,jh_1]\\ & & \quad +[ji_1,h_1,jh_1,i_1]+[ji_1,h_1,i_1,jh_1])\\ & = & -(ab-cd)^2=-1.\end{array}$$

On the 378 images of the second quadruple, there are four orbits, of lengths 6, 60, 120, and 192, represented respectively by

$$[h_1,jh_1,i_1,ji_1],[h_1,jh_1,j_1,jj_1],[h_0,jh_0,i_0,ji_0],[h_0,jh_0,h_1,jh_1].$$

Of these, the third is obviously fixed by the $\mathrm{SL}(2,\mathbb{C})$, and the other three are straightforward calculations. For example, the last case is

$$\begin{array}{rcl}[h_0,jh_0,h_1,jh_1] & \mapsto & [h_0,jh_0,ah_1+bji_1,djh_1+ci_1]\\ & = & ad[h_0,jh_0,h_1,jh_1]+bc[h_0,jh_0,ji_1,i_1]\\ & = & (ad-bc)[h_0,jh_0,h_1,jh_1]\end{array}$$

by applying a suitable Weyl group element (for example, $H_1\mapsto jI_1\mapsto -H_1, I_1\mapsto jH_1\mapsto -I_1$) to the second term. Similarly,

$$\begin{array}{rcl}[h_1,jh_1,i_1,ji_1] & \mapsto & [ah_1+bji_1,djh_1-ci_1,ai_1+bjh_1,dji_1-ch_1]\\ & = & (ad)^2[h_1,jh_1,i_1,ji_1]+(bc)^2[ji_1,i_1,jh_1,h_1]\\ & & +abcd([h_1,i_1,jh_1,ji_1]+[ji_1,jh_1,i_1,h_1]\\ & & \quad +[h_1,jh_1,jh_1,h_1]+[ji_1,i_1,i_1,ji_1])\\ & = & \frac{1}{2}(ad-bc)^2=\frac{1}{2},\\ [h_1,jh_1,j_1,jj_1] & \mapsto & [ah_1+bji_1,djh_1+ci_1,dj_1-cjk_1,ajj_1-bk_1]\\ & = & (ad)^2[h_1,jh_1,j_1,jj_1]+(bc)^2[ji_1,i_1,jk_1,k_1]\\ & & +abcd([h_1,jh_1,jk_1,k_1]-[h_1,i_1,j_1,k_1]\\ & & \quad +[ji_1,i_1,j_1,jj_1]-[ji_1,jh_1,jk_1,jj_1])\\ & = & (ad-bc)^2/2.\end{array}$$

On the $2\times 315=630$ images of the third quadruple, there are just three orbits, of lengths 30, 120, and 480. These are represented respectively by

$$[h_1,i_1,j_1,k_1],[h_0,i_0,j_0,k_0],[h_3,j_3,j_2,k_2].$$

Again, the second is obviously fixed, and the other two are straightforward calculations:

$$\begin{array}{rcl}[h_1,i_1,j_1,k_1] & \mapsto & [ah_1+bji_1,ai_1+bjh_1,dj_1-cjk_1,dk_1-cjj_1]\\ & = & (ad)^2[h_1,i_1,j_1,k_1]+(bc)^2[jh_1,ji_1,jj_1,jk_1]\\ & & -abcd([h_1,jh_1,j_1,jj_1]+[h_1,jh_1,jk_1,k_1]\\ & & \quad +[ji_1,i_1,j_1,jj_1]+[ji_1,i_1,jk_1,k_1]\\ & = & (ad)^2-2abcd+(bc)^2=1,\\ [h_3,j_3,j_2,k_2] & \mapsto & [h_3,j_3,aj_2+bjh_2,dk_2-cji_2]\\ & = & ad[h_3,j_3,j_2,k_2]-bc[h_3,j_3,jh_2,ji_2]\\ & = & (ad-bc)[h_3,j_3,j_2,k_2]\\ & = & [h_3,j_3,j_2,k_2].\end{array}$$

In fact, in order to prove that the fundamental $\mathrm{SL}(2,\mathbb{C})$ preserves the quadrilinear form, we must also check that it preserves all the zero values. Now the only way one of these zero values could not be preserved, is if one of the non-zero values we have just studied gets added to it. This means that the only zero values we need to consider are those which were actually used in the above calculations. These are as follows:

$$[h_1, h_1, jh_1, i_1], [h_0, jh_0, h_1, i_1], [h_1, h_1, i_1, i_1],$$
$$[h_1, h_1, jh_1, i_1], [h_1, i_1, jk_1, jj_1], [h_1, i_1, jk_1, k_1].$$

Thus we have a little more calculation to do of the same kind we've already done: this is left as an exercise for the reader.

Having completed these calculations, we have proved from first principles that the group elements defined above preserve the quadrilinear form defined in this section.

5. Some subgroups

If we take a sub-root system of E_7, then the subgroup generated by the corresponding root SU(2) subgroups is often of interest. For example, the subsystem spanned by all the roots of the shape $\pm(2, 0^6)$ gives rise to a central product of seven copies of SU(2), in which the centre is reduced from 2^7 to 2^3. This subgroup stabilizes the seven 4-dimensional spaces $\langle h_t, i_t, j_t, k_t\rangle$, for $t \in \mathbb{F}_7$, and the Weyl group induces a transitive permutation action of $\mathrm{PSL}_3(2)$ on these seven subspaces, as well as (a different action) on the seven SU(2) factors of the group.

This subsystem extends to a system of type $A_1A_1A_1D_4$ by adjoining all the roots $(\pm1, 0, 0, \pm1, 0, \pm1, \pm1)$ with 0s on coordinates $1, 2, 4$. This subsystem group fixes the quaternionic 4-space $\langle h_0, i_0, j_0, k_0\rangle$, and is normalized by a triality element which acts as

$$(i, j, k)(1, 2, 4)(3, 6, 5).$$

We may extend further to A_1D_6, consisting of all the roots equal or perpendicular to $\pm(2, 0, 0, 0, 0, 0, 0)$. The corresponding subsystem group splits the space into the part with suffices $1, 2, 4$, on which the A_1 and D_6 both act naturally, that is as $\mathrm{SU}(2) \otimes \mathrm{O}(12)$; and the part with suffices $0, 3, 5, 6$, on which the A_1 acts trivially and the D_6 acts in its spin representation.

Another maximal rank subsystem which is of interest is A_7. While there is no particularly symmetrical copy of A_7 to choose, this maximal rank subgroup is the one used by Cooperstein [3] in his construction of E_7.

Perhaps the most interesting subsystem is E_6, which may be spanned by all the roots perpendicular to $(0, 1, 1, 0, 1, 0, 0)$. Thus it is clear that the stabilizer of the quaternionic 1-space $\langle h_0\rangle$ corresponding to this vector is a copy of (the simply-connected compact real form of) E_6, extended by a 1-dimensional torus, and a duality map induced by the central involution in the Weyl group. To see this in a 'classical' way, consider all the terms in the quadrilinear form which involve h_0 once only, and remove this factor h_0 from them. We obtain a symmetric trilinear form in 27 variables which is the polarized form of Dickson's cubic form for E_6. An explicit correspondence between our coordinates and Dickson's is given by the following table, where each quaternionic coordinate q is split into its complex and imaginary parts as $q = q' + q''j$ with $q', q'' \in \mathbb{C}$. The entries in the body of the table are $z_{rs} = -z_{sr}$.

	r	1	2	3	4	5	6
s	$y_s\backslash x_r$	H_1''	H_2''	H_3''	H_4''	H_5''	H_6''
1	K_3''		K_4'	I_0'	$-J_2'$	J_6'	$-I_5'$
2	I_6''	$-K_4'$		K_5'	I_1'	$-J_3'$	J_0'
3	$-J_1''$	$-I_0'$	$-K_5'$		K_6'	I_2'	$-J_4'$
4	J_5''	J_2'	$-I_1'$	$-K_6'$		K_0'	I_3'
5	$-I_4''$	$-J_6'$	J_3'	$-I_2'$	$-K_0'$		K_1'
6	$-K_2''$	I_5'	$-J_0'$	J_4'	$-I_3'$	$-K_1'$	

The 45 terms of Dickson's cubic form are $x_r y_r z_{rs}$ and $z_{rs}z_{tu}z_{vw}$, where $rstuvw$ is an even permutation of 123456. In our notation, the latter are $I_0'J_0'K_0'$, $I_3'J_6'K_5'$, $-I_5'J_3'K_6'$ and the images under the triality element $(1,2,4)(3,6,5)(I,J,K)$ of

$$I_0'I_1'K_1', I_0'I_3'J_3', J_2'K_1'K_5', -I_1'I_2'I_5'.$$

The former are the images under triality of the following 10 terms:

$$\begin{array}{c} I_0'H_1''J_1'', I_0'H_3''K_3'', I_3'H_6''J_5'', -I_5'H_6''K_3'', \\ -I_1'H_2''J_5'', -I_2'H_5''J_1'', -I_5'H_1''K_2'', K_1'H_5''K_2'', J_2'H_1''J_5'', K_5'H_2''J_1''. \end{array}$$

Finally we remark that E_6 has a subgroup of type F_4, which in our representation fixes a quaternionic 2-space, such as $\langle h_0, i_0+j_0+k_0\rangle$.

6. The split real form and finite groups of type E_7

To construct the split real form we take the complexification as defined above, and simply restrict the matrix entries to lie in $\mathbb{R}$, so that the root groups become $\mathrm{SL}(2,\mathbb{R})$. This now defines an action of the Weyl group and a root $\mathrm{SL}(2,\mathbb{R})$ on a 56-dimensional real vector space, where the typical vector is written as

$$(H_0', H_0'', I_0', \ldots, K_6', K_6'').$$

Hence we have generators for the (simply-connected) split real form of E_7. To be precise, the action of generators for our standard root $\mathrm{SL}(2,\mathbb{R})$ is given by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

acting on the real 2-vectors

$$(H_1', I_1''), (I_1', H_1''), (H_2'J_2''), (J_2', H_2''), (H_4', K_4''), (K_4', H_4'')$$

and as

$$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}, \begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

acting on the real 2-vectors

$$(J_1', K_1''), (K_1', J_1''), (K_2'I_2''), (I_2', K_2''), (I_4', J_4''), (K_4', H_4'').$$

The same matrices can be interpreted over any field F whatever, and give generators for the (simply-connected) split real form of $E_7(F)$. (Over finite fields all forms of E_7 are split.)

The symplectic form is defined by $(x', jx') = -(jx', x') = 1$, for each of the 28 possible values of x, and all other inner products are 0. The quadrilinear form needs no change to its definition in any odd characteristic. In characteristic 2, as usual, the situation is more complicated. However, we can just define $[h_0, i_0, j_0, k_0] = 1$, together with its images under the Weyl group, and let all the other terms be 0. Then the arguments of Section 4 go through also in characteristic 2.

The same argument as in the complex case shows that the centralizer of the 2-space $\langle h_0, jh_0\rangle$ is (the simply-connected version of) $E_6(q)$. The whole stabiliser of the 1-space $\langle h_0\rangle$, therefore, is generated by $E_6(q)$, together with one more dimension of torus, acting as $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ on (h_0, jh_0), and 27 root groups which add multiples of 27 coordinates (these are exactly the coordinates corresponding to the 27 variables of Dickson's cubic form, as specified above) onto jh_0. Thus the stabiliser of $\langle h_0\rangle$ (a 1-space over $\mathbb{F}_q$) is a group of shape $q^{27}{:}(C_{q-1} \times E_6(q))$, in the case when $q \equiv 2 \bmod 3$, or $2^{27}{:}3.(C_{(q-1)/3} \times E_6(q)).3$ if $q \equiv 1 \bmod 3$.

Now we need to count the images of $\langle h_0\rangle$. The Weyl group maps h_0 to one of the 56 coordinate 1-spaces, and each of these 56 spaces is in an orbit under the stabiliser of $\langle h_0\rangle$ of size a power of q. It is possible to show, with a sufficient amount of tedious but not difficult work, that these powers of q are as follows:

$$\begin{array}{l}0,1,2,3,4,5,6,7,8,9,10,11,12,13,\\ 5,6,7,8,9,10,11,12,13,14,15,16,17,18,\\ 9,10,11,12,13,14,15,16,17,18,19,20,21,22,\\ 14,15,16,17,18,19,20,21,22,23,24,25,26,27\end{array}$$

and that therefore the number of images of $\langle h_0\rangle$ is

$$(1+q^5)(1+q^9)(q^{14}-1)/(q-1).$$

It follows that the order of the group is

$$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1).$$

In odd characteristic, there is a centre of order 2, so the simple group $E_7(q)$ has order half this. In characteristic 2, the group is already simple as the centre is trivial.

7. Further remarks

7.1. **The labelling by pairs of 8 points.** The 28 objects permuted by the Weyl group of type E_7 are traditionally labelled by the unordered pairs from 8 points, so we here label our 28 coordinates thus. We label the 8 points by the projective line $\mathbb{F}_7 \cup \{\infty\}$ and then label

$$h_0 = \{\infty, 0\}, i_0 = \{1,3\}, j_0 = \{2,6\}, k_0 = \{4,5\}$$

and let the map $t \mapsto t+1$ on subscripts also act as $t \mapsto t+1$ on the projective line. There is a subgroup S_8 of the Weyl group which acts by permuting $\{\infty, 0, 1, 2, 3, 4, 5, 6, 7\}$. Fixing $\infty, 0$ gives a subgroup S_6 which has orbits of sizes $6+6+15$ on the 27 remaining objects, and shows clearly the correspondence with Dickson's cubic form for E_6.

7.2. **The second copy of the quaternions.** The reason for labelling the 7 sets of four quaternionic coordinates with the letters H, I, J, K is that these behave in many ways like the quaternions $1, i, j, k$. Thus we define products of these as in the quaternion group: $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$ and $I^2 = J^2 = K^2 = -H$, with H acting as the identity element. Analogous to the element $\omega = \frac{1}{2}(-1+i+j+k)$ of order 3 we have $\Omega = \frac{1}{2}(-H+I+J+K)$.

Now consider our 28-dimensional quaternionic space as a 7-dimensional 'space' over the tensor product of these two copies of the quaternions. A 'vector' may be represented $(X_0, X_1, X_2, X_3, X_4, X_5, X_6)$, and the actions of some of the elements given above are as follows:

- the element $t \mapsto t+1$ acts as $(X_t) \mapsto (X_{t+1})$;
- the triality element is $(X_t) \mapsto (X_{2t}^{\Omega})$;
- $(X_t) \mapsto (-X_0^{I-J}, X_2^{\overline{\Omega}}, X_1^{\Omega}, X_6^{I-K}, X_4, X_5^{I-K}, X_3^{I-K})$.

In order to express the action of our typical element of a fundamental SU(2), we write $z = a + bi$, $wj = cj + dk$, where $a, b, c, d \in \mathbb{R}$. It now acts as

$$\begin{array}{rcl} X_1 & \mapsto & X_1a - IX_1Ibi - KX_1Jcj - JX_1Kdk \\ X_2 & \mapsto & X_2a - JX_2Jbi - IX_2Kcj - KX_2Idk \\ X_4 & \mapsto & X_4a - KX_4Kbi - JX_4Icj - IX_4Jdk \end{array}$$

In effect, the quaternion group generated by right-multiplication by i and j has been replaced by a different quaternion group in each of the three coordinates. The non-trivial element of the Weyl group in this SU(2) has $a = b = d = 0$, $c = 1$, that is

- $(X_1, X_2, X_4) \mapsto (-KX_1Jj, -IX_2Kj, -JX_4Ij)$.

References

[1] M. Aschbacher, Some multilinear forms with large isometry groups, *Geom. Dedicata* **25** (1988), 417–465.

[2] R. B. Brown, Groups of type E_7, *J. Reine Angew. Math.* **236** (1969), 79–102.

[3] B. N. Cooperstein, The fifty-six dimensional module for E_7. I. A four-form for E_7, *J. Algebra* **173** (1995), 361–389.

[4] L. E. Dickson, A new system of simple groups, *Math. Ann.* **60** (1905), 137–150.

[5] L. E. Dickson, A class of groups in an arbitrary realm connected with the configuration of the 27 lines on a cubic surface, *Quart. J. Pure Appl. Math.* **33** (1901),145–173.

[6] L. E. Dickson, A class of groups in an arbitrary realm connected with the configuration of the 27 lines on a cubic surface (second paper), *Quart. J. Pure Appl. Math.* **33** (1908),205–209.

[7] R. A. Wilson, *The finite simple groups*, Springer GTM 251, 2009.

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