

The Alperin weight conjecture and Dade's conjecture for the simple group J_4

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Abstract

We construct faithful permutation representations of maximal 2-local subgroups and classify the radical chains of the Janko simple group J_4 ; hence we verify the Alperin weight conjecture and the Dade reductive conjecture for J_4 .

1 Introduction

The program of deciding Dade's reductive conjecture [12] for the sporadic groups has made very substantial progress: it has been verified for all of the sporadic simple groups except Fi'_{24} , J_4 , B and M .

The use of computer algebra systems, namely MAGMA [9] and GAP [14], to study permutation (or in some cases matrix) representations of the groups has been a central step of the program. Since the smallest faithful permutation representation of J_4 has degree 173067389, it is difficult to verify the conjecture directly. However, from the classification of maximal subgroups of J_4 [16], we know that the normalizer of each radical 2- and 3-subgroup of J_4 is a subgroup of one of precisely four maximal 2-local subgroups. Thus we can classify radical chains in these four maximal subgroups without performing any calculation in J_4 .

In this paper, we construct faithful permutation representations for each maximal 2-local subgroup, classify radical chains, and hence verify the Alperin weight conjecture and Dade reductive conjecture for J_4 .

Let G be a finite group, p a prime and B a p -block of G . Alperin [1] conjectured that the number of B -weights equals the number of irreducible Brauer characters of B . Dade [11] generalized the Knörr–Robinson version of the Alperin weight conjecture and presented his ordinary conjecture exhibiting the number of ordinary irreducible characters of a fixed defect in B in terms of an alternating sum of related values for p -blocks of some p -local subgroups of G . Dade [12] announced that his reductive conjecture needs only to be verified for finite non-abelian simple groups; in addition,

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if a finite group has a trivial Schur multiplier and trivial outer automorphism group, then the ordinary conjecture is equivalent to the reductive conjecture.

The paper is organized as follows. In Section 2, we fix notation, state the conjectures in detail and state two lemmas. In Section 3 we explain how to construct faithful permutation representations of the four maximal 2-local subgroups. In Section 4, we recall the modified local strategy [4] and [5]; we also explain how we applied it to determine the radical subgroups of each maximal subgroup, and how to fuse the radical subgroups in J_4 . In Section 5, we use the list of radical subgroups of J_4 given by [20] to verify the Alperin weight conjecture. In Section 6, we do some cancellations in the alternating sum of Dade's conjecture, and then determine radical chains (up to conjugacy) and their local structures, and finally we verify the ordinary conjecture of Dade for J_4 .

2 Dade's ordinary conjecture

Let R be a p -subgroup of a finite group G . Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p -subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\text{Irr}(G)$ the set of all irreducible ordinary characters of G , and let $\text{Blk}(G)$ be the set of p -blocks, $B \in \text{Blk}(G)$ and $\varphi \in \text{Irr}(N(R)/R)$. The pair (R, φ) is called a *B-weight* if $d(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $d(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ is the p -defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ . A weight is always identified with its G -conjugates. Let $\mathcal{W}(B)$ be the number of B -weights, and $\ell(B)$ the number of irreducible Brauer characters of B . Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \text{Blk}(G)$.

Given a p -subgroup chain

$$C : P_0 < P_1 < \cdots < P_n \tag{2.1}$$

of G , define $|C| = n$, $C_k : P_0 < P_1 < \cdots < P_k$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \cdots \cap N(P_n). \tag{2.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions:

- (a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$ for $1 \leq k \leq n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p -chains of G .

Let $k(N_G(C), B, d)$ be the number of characters ψ in $\text{Irr}(N_G(C))$ such that $d(\psi) = d$ and $B(\psi)^G = B$. In the notation above the Dade ordinary conjecture is stated as follows.

Dade's Ordinary Conjecture [11]. *If $O_p(G) = 1$ and B is a p -block of G with defect group $D(B) \neq 1$, then for any integer $d \geq 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N_G(C), B, d) = 0$$

where \mathcal{R}/G is a set of representatives for the G -orbits of \mathcal{R} .

Let G be the Janko simple group J_4 . Then its Schur multiplier and outer automorphism group are both trivial, so by [12], Dade's ordinary conjecture is equivalent to his reductive conjecture.

In Section 6, we shall use the following lemmas.

Lemma 2.1 *Let $\sigma : O_p(G) < P_1 < \dots < P_{m-1} < Q = P_m < P_{m+1} < \dots < P_\ell$ be a fixed radical p -chain of a finite group G , where $1 \leq m < \ell$. Suppose*

$$\sigma' : O_p(G) < P_1 < \dots < P_{m-1} < P_{m+1} < \dots < P_\ell$$

is also a radical p -chain such that $N_G(\sigma) = N_G(\sigma')$. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell - 1)$ -th subchain $C_{\ell-1}$ is conjugate to σ' in G , and $\mathcal{R}^0(\sigma, Q)$ the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓ -th subchain C_ℓ is conjugate to σ in G . Then the map g sending any $O_p(G) < P_1 < \dots < P_{m-1} < P_{m+1} < \dots < P_\ell < \dots$ in $\mathcal{R}^-(\sigma, Q)$ to $O_p(G) < P_1 < \dots < P_{m-1} < Q < P_{m+1} < \dots < P_\ell < \dots$ induces a bijection, denoted again by g , from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| - 1$ and $N_G(C) = N_G(g(C))$.

PROOF: Straightforward. □

Lemma 2.2 *If Q is a p -subgroup of a finite group G , then there is a radical p -subgroup R such that*

$$Q \leq R \quad \text{and} \quad N_G(Q) \leq N_G(R).$$

PROOF: This follows by [2, Lemma 2.1]. □

3 Construction of permutation representations of maximal 2-local subgroups

We will follow the notation of [10]. In particular, $p^{1+2\gamma} = p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p or type $+$ according as p is odd or even. If X and Y are groups, we use $X \cdot Y$ and $X : Y$ to denote a nonsplit extension and a split extension of X by Y , respectively. Given a positive integer n , we use p^n to denote the elementary abelian group of order p^n , n to denote the cyclic group of order n , and D_{2n} to denote the dihedral group of order $2n$.

The four maximal 2-local subgroups of J_4 which we wish to construct are $2^{11} : M_{24}$, $2^{10} : L_5(2)$, $2^{3+12} \cdot (S_5 \times L_3(2))$ and $2^{1+12} \cdot 3 \cdot M_{22} : 2$. The first two of these are easy to construct abstractly as affine groups. The first can be written as 12×12 matrices over $GF(2)$, acting either with a fixed vector or with a fixed hyperplane. These representations can be easily obtained from a submodule or quotient module (respectively) of the restriction of the 112-dimensional representation of J_4 . These give rise naturally to permutation representations on $759 + 1288 + 2048$ non-zero vectors in the one case, or $1 + 1518 + 2576$ in the second. Any of the three faithful representations, of degrees 2048, 1518, and 2576, can then be used to generate the character table of the group.

Similarly, the group $2^{10}:L_5(2)$ has two natural affine representations, in which the orbits of non-zero vectors are either $155 + 868 + 1024$ or $1 + 310 + 1736$ in length, and again any of the three faithful representations, of degree 1024, 310 and 1736, can be used as a starting point for the calculations.

The other two maximal 2-local subgroups of J_4 are harder to construct. They can be obtained as subgroups of J_4 using the words in the standard generators of J_4 given in [7]. This however limits us to two particular representations of the groups, given by restricting the representations of J_4 in dimension 112 over $GF(2)$ or dimension 1333 over $GF(11)$. For the purposes of calculating the character tables, we want to obtain faithful permutation representations on a reasonably small number of points.

On the other hand, of course, the point stabilizer in a faithful representation cannot contain any non-trivial normal subgroup, and this puts severe constraints on which subgroups we can use as a point stabilizer. For example, in the case $2^{1+12}\cdot 3\cdot M_{22}:2$, if we are to avoid the central involution, the point stabilizer can have at most a subgroup of order 2^6 in the normal 2^{1+12} . Moreover, such a subgroup must be invariant under a reasonable subgroup of $6\cdot M_{22}$, which must again split off the central involution. The biggest subgroup we could identify as a potential point stabilizer was a group of shape $2^6:3\cdot A_6$, which we found as the subgroup generated by certain words in the given generators of $2^{1+12}\cdot 3\cdot M_{22}:2$.

This subgroup has index $2^7\cdot 2.77.16 = 630784$. We obtained the corresponding permutation representation on 630784 points by calculating the subgroup explicitly in the 112-dimensional representation over $GF(2)$, and using MAGMA to calculate the permutation action on the cosets. We then found that in fact there was a block system of 315392 blocks of size 2, such that the action on the blocks was still faithful. This representation was then used as input to the next stage of calculation.

In the remaining case, we found a subgroup of shape $2^{1+8}(5 \times S_4)$ as a potential point stabilizer in $2^{3+12}(S_5 \times L_3(2))$. In this case we used the 1333-dimensional $GF(11)$ -representation of J_4 , and restricted to a 1120-dimensional submodule on which the maximal 2-local subgroup acts faithfully and irreducibly. Restricting further to the potential point stabilizer, we found a 15-dimensional invariant subspace. Taking the permutation action of our maximal 2-local subgroup on the 61440 images of this 15-space gave us the desired faithful permutation representation.

4 A local subgroup strategy and fusions

Kleidman and Wilson [16] classified the maximal subgroups of J_4 . From this, we know that there are 4 maximal 2-local subgroups up to conjugacy. In addition, each radical 2- and 3-subgroup R of J_4 is radical in one of the subgroups M and further $N_{J_4}(R) = N_M(R)$. The radical 2-subgroups of J_4 are classified by [20].

In [4] and [5], a (modified) local strategy was developed to classify the radical p -subgroups R . We review this method here.

Let $Q = O_p(M)$, so that $Q \leq R$. Choose a subgroup X of M . Using MAGMA, we explicitly compute the coset action of M on the cosets of X in M ; we obtain a group W representing this action, a group homomorphism f from M to W , and the kernel K of f . For a suitable X , we have $K = Q$ and the degree of the action of W on the cosets

is much smaller than that of M . We can now directly classify the radical p -subgroup classes of W , and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M .

After applying the strategy, we list the radical subgroups of each M and do the fusions using local structures of $N_M(R)$ and Lemma 2.2. Some details are given in the proof of Lemma 6.1.

The computations reported in this paper were carried out using MAGMA V2.9-9 on a Sun UltraSPARC Enterprise 4000 server.

5 Weights

Let $\mathcal{R}_0(G, p)$ be a set of representatives for conjugacy classes of radical p -subgroups of G . For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \mathcal{R}_0(G, p)$ if $x^{-1}Hx \in \mathcal{R}_0(G, p)$ for some $x \in G$.

Let G be the Janko simple group J_4 . Then

$$|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43,$$

and we may suppose $p \in \{2, 3, 11\}$, since both conjectures hold for a block with a cyclic defect group by [11, Theorem 7.1]. If $p = 11$, then by [8, Proposition 1.3], a Sylow subgroup of G is a trivial intersection group, so that Dade's ordinary conjecture and Alperin's weight conjecture follow by [13]. Moreover, Uno's ordinary conjecture, which is refinement of Dade's ordinary conjecture also holds for J_4 by [3]. Thus we may suppose $p = 2$ or 3 .

We denote by $\text{Irr}^0(H)$ the set of ordinary irreducible characters of p -defect 0 of a finite group H and by $d(H)$ the number $\log_p(|H|_p)$. Given $R \in \mathcal{R}_0(G, p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p -block of G , then (c.f. (4.1) of [4])

$$\mathcal{W}(B_0) = \sum_R |\text{Irr}^0(N/C(R)R)|, \quad (5.1)$$

where R runs over the set $\mathcal{R}_0(G, p)$ such that $d(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\text{Irr}^0(N/C(R)R)|$.

In Table 1, we recall the classification from [20] of the radical 2-subgroups of $G = J_4$. Suppose $p = 3$. As shown in [16, Section 3],

$$\mathcal{R}_0(G, 3) = \{1, 3, 3^2, 3_+^{1+2}\},$$

where $3 = Z(3_+^{1+2})$. In addition, $C(3) = 6.M_{22}$, $C(3^2) = 3^2 \times 2^3$, $C(3_+^{1+2}) = 6$ and

$$N(R) = \begin{cases} 6.M_{22}:2 \leq 2_+^{1+12}.3.M_{22}:2 & \text{if } R = 3, \\ (3^2:2 \times 2^3).S_4 \leq 2^{11}.M_{24} & \text{if } R = 3^2, \\ (2 \times 3_+^{1+2}:8):2 \leq 2_+^{1+12}.3.M_{22}:2 & \text{if } R = 3_+^{1+2}. \end{cases}$$

Lemma 5.1 *Let $G = J_4$ and $B_0 = B_0(G)$, and let $\text{Blk}^+(G, p)$ be the set of p -blocks with a non-trivial defect group and $\text{Irr}^+(G)$ the characters of $\text{Irr}(G)$ with positive p -defect. If a defect group $D(B)$ of B is cyclic, then $\text{Irr}(B)$ is given by [17, p. 326].*

- (a) If $p = 3$, then $\text{Blk}^+(G, p) = \{B_i \mid 0 \leq i \leq 6\}$ such that $D(B_1) \simeq 3_+^{1+2}$, $D(B_2) \simeq 3^2$ and $D(B_i) \simeq 3$ for $3 \leq i \leq 6$. In the notation of [10, p. 188],

$$\text{Irr}(B) = \{\chi_2, \chi_3, \chi_{12}, \chi_{13}, \chi_{17}, \chi_{18}, \chi_{22}, \chi_{23}, \chi_{24}, \chi_{26}, \chi_{38}, \chi_{39}, \chi_{44}, \chi_{50}\},$$

$$\text{Irr}(B_2) = \{\chi_{14}, \chi_{21}, \chi_{25}, \chi_{27}, \chi_{28}, \chi_{30}, \chi_{31}, \chi_{35}, \chi_{41}\}, \text{ and}$$

$$\text{Irr}(B_0) = \text{Irr}^+(G) \setminus (\cup_{i=1}^6 \text{Irr}(B_i)).$$

Moreover, $\ell(B_0) = \ell(B_1) = 9$, $\ell(B_2) = 5$, $\ell(B_i) = 2$ for $3 \leq i \leq 5$ and $\ell(B_6) = 1$.

R	$C(R)$	$N(R)$	$ \text{Irr}^0(N/C(R)R) $
2^{10}	2^{10}	$2^{10}:L_5(2)$	1
2^{11}	2^{11}	$2^{11}:M_{24}$	0
2^{1+12}	2	$2^{1+12}.(3.M_{22}):2$	1
$2^{10}:2^4$	2^4	$2^{10}:2^4.L_4(2)$	1
$2^{11}:2^4$	2^6	$2^{11}:2^4.A_8$	1
2^{3+12}	2^3	$2^{3+12}.(S_5 \times L_3(2))$	0
$2^{3+12}.2$	2^3	$2^{3+12}.2.(S_3 \times L_3(2))$	1
$2^{1+12}.2^3$	2	$2^{1+12}.2^3.(S_3 \times L_3(2))$	1
$2^3.2^{6+8}$	2^3	$2^3.2^{6+8}.(S_3 \times L_3(2))$	1
$2^2.2^{5+10}$	2^2	$2^2.2^{5+10}.(S_3 \times S_5)$	0
$2^{1+12}.2^4$	2	$2^{1+12}.2^4.3.S_6$	1
$2^{3+12}.2^2$	2	$2^{2+12}.2^2.(S_3 \times S_5)$	0
$2^{10}.2^{3+4}$	2	$2^{10}.2^{3+4}.L_3(2)$	1
$2^{3+12}.D_8$	2^3	$2^{3+12}.D_8.L_3(2)$	1
$2^{3+12}.2^3$	2^2	$2^{3+12}.2^3.(S_3 \times S_3)$	1
$2^{1+12}.2^{2+3}$	2	$2^{1+12}.2^{2+3}.(S_3 \times S_3)$	1
$2^{11}.2^{1+6}$	2^3	$2^{11}.2^{1+6}.L_3(2)$	1
$2^{1+12}.2^5$	2	$2^{1+12}.2^5.S_5$	0
$2^{1+12}.2^2.2^4$	2	$2^{1+12}.2^2.2^4.(S_3 \times S_3)$	1
$2^{1+12}.2.2^{2+3}$	2	$2^{1+12}.2.2^{2+3}.(S_3 \times S_3)$	1
$2^{3+12}.2^4$	2	$2^{3+12}.2^4.(S_3 \times S_3)$	1
$2^{1+12}.2^3.2^3$	2	$2^{1+12}.2^3.2^3.S_3$	1
$2^{3+12}.2^3.2^2$	2^2	$2^{3+12}.2^3.2^2.S_3$	1
$2^{11}.2^2.2^3.2^4$	2	$2^{11}.2^2.2^3.2^4.S_3$	1
$2^{1+12}.2^{2+5}$	2	$2^{1+12}.2^{2+5}.S_3$	1
$2^{1+12}.2.2^3.2^3$	2	$2^{1+12}.2.2^3.2^3.S_3$	1
S	2	S	1

Table 1: Non-trivial radical 2-subgroups of J_4

(b) If $p = 2$, then $\text{Blk}^+(G, 2) = \{B_0\}$ and so $\text{Irr}(B_0) = \text{Irr}^+(G)$. Moreover, $\ell(B_0) = 22$.

PROOF: If $B \in \text{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\text{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p' -group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By [17, p. 326], we may suppose D is non-cyclic. Thus $D = 3^2$ or 3_+^{1+2} . In each case, $N(D)$ has one orbit on the non-trivial character of $\text{Irr}^0(C(D)D/D)$ with $N(\theta)/C(D)D$ a $3'$ -group.

Using the method of central characters, $\text{Irr}(B)$ is as above. If $D(B)$ is cyclic, then $\ell(B)$ is given by [17, p. 326].

If $p = 3$ and $B = B_1$ or B_2 , then the non-trivial elements of $D(B)$ are of type $3A$, and $C_G(3A) = 6.M_{22}$. It follows by [18, Theorem 5.4.13] that

$$k(B) = \ell(B) + \sum_{b \in \text{Blk}(6.M_{22}, B)} \ell(b),$$

where $\text{Blk}(6.M_{22}, B) = \{b \in \text{Blk}(6.M_{22}) : b^G = B\}$.

If $B = B_2$, then $D(B_2) =_G 3^2$, $\text{Blk}(6.M_{22}, B) = \{b_2, b'_2\}$ and by [17, p. 78], $\ell(b_2) = \ell(b'_2) = 2$ and so $\ell(B_2) = 5$.

If $B = B_1$, then $D(B_1) =_G 3_+^{1+2}$, $\text{Blk}(6.M_{22}, B) = \{b\}$ and moreover, $\ell(b) = \ell(b_1)$ for a unique block of $2.M_{22}$ with $D(b_1) \simeq 3^2$. Now $2.M_{22}$ has exactly one class x of elements of order 3 and $C_{2.M_{22}}(x) = 3^2 \times 2^3$. It follows by [18, Theorem 5.4.13] again that $\ell(b_1) = k(b_1) - 1 = 6 - 1 = 5$, so that $\ell(B_1) = 14 - 5 = 9$.

If $\ell_p(G)$ is the number of p -regular G -conjugacy classes, then $\ell_3(G) = 43$ and $\ell_2(G) = 25$. Thus $\ell(B_0)$ can be calculated by the following equation due to Brauer:

$$\ell_p(G) = \sum_{B \in \text{Blk}^+(G, p)} \ell(B) + |\text{Irr}^0(G)|.$$

This completes the proof. □

Theorem 5.2 *Let $G = J_4$ and let B be a p -block of G with a non-cyclic defect group. Then the number of B -weights is the number of irreducible Brauer characters of B .*

PROOF: We may suppose $p = 2$ and 3 . If $p = 2$ and $B = B_0$, then it follows by Lemma 5.1, Table 1 and (5.1).

Suppose $p = 3$. Since $\text{Irr}(N(3^{1+2})/C(3^{1+2}))$ has seven irreducible characters and since $\text{Irr}(N(3^{1+2})/3^{1+2})$ has 14 characters, it follows that $B \in \{B_0, B_1\}$ has 7-weights of the form $(3^{1+2}, \varphi)$. If $(3^2, b)$ is a Brauer B -subgroup, then $\text{Irr}^0(N(3^2)/3^2)$ has exactly two characters covering the canonical character of b , so $\text{Irr}^0(N(3^2)/C(3^2))$ has two characters, B has 2-weights of the form $(3^2, \varphi)$ and B has no weight of the form $(3, \varphi)$.

$\text{Irr}^0(N(3^2)/3^2)$ has 9 characters, hence B_2 has 5-weights of the form $(3^2, \varphi)$. □

6 Radical chains

Let $G = J_4$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. We will do some cancellations in the alternating sum of Dade's conjecture. We first list some radical p -chains $C(i)$ and their normalizers for certain integers i , then reduce the proof of the conjecture to the subfamily $\mathcal{R}^0(G)$ of $\mathcal{R}(G)$, where $\mathcal{R}^0(G)$ is the union of G -orbits of all $C(i)$. The subgroups of the 2-chains in Table 3 are given either by Table 1 or in the proof of Lemma 6.1.

Lemma 6.1 *Let $\mathcal{R}^0(G)$ be the G -invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leq i \leq 4\} & \text{with } C(i) \text{ given in Table 2 if } p = 3, \\ \{C(i) : 1 \leq i \leq 16\} & \text{with } C(i) \text{ given in Table 3 if } p = 2. \end{cases}$$

Then

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} k(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} k(N(C), B_0, d) \quad (6.1)$$

for all integers $d \geq 0$.

C		$N(C)$
$C(1)$	1	J_4
$C(2)$	$1 < 3$	$6.M_{22}:2$
$C(3)$	$1 < 3 < 3^2$	$(3^2 \times 2^3).(2 \times S_3)$
$C(4)$	$1 < 3^2$	$(3^2:2 \times 2^3).S_4$

Table 2: Some radical 3-chains of J_4

PROOF: Let $C \in \mathcal{R}(G)$ be given by (2.1), so that we may suppose $P_1 \in \mathcal{R}_0(G, p)$.

Case (1) Suppose $p = 3$. Let $C' : 1 < 3 < 3^{1+2}$ and $g(C') : 1 < 3^{1+2}$. Then $N(C') = N(g(C')) = N(3^{1+2})$,

$$k(N(C'), B, d) = k(N(g(C')), B, d), \quad (6.2)$$

and we may suppose $C \neq_G C'$ and $g(C')$. Similarly, let $C' : 1 < 3^2 < 3^{1+2}$ and $g(C') : 1 < 3 < 3^2 < 3^{1+2}$. Then $N(C') = N(g(C')) =_G 3^{1+2}.2^3$ and we may suppose $C \neq_G C'$ and $g(C')$. Thus $C =_G C(i)$ for $1 \leq i \leq 4$.

Case (2) Suppose $p = 2$. Let $M_1 = 2^{11}:M_{24}$, $M_2 = 2^{1+12}.(3.M_{22}):2$, $M_3 = 2^{10}:L_5(2)$ and $M_4 = 2^{3+12}.(S_5 \times L_3(2))$ be maximal subgroups of $G = J_4$. For each $R \in \mathcal{R}_0(G, 2)$, we may suppose $R \in \mathcal{R}_0(M_i)$ such that $N_G(R) \leq N_{M_i}(R)$ for some i .

We first classify the radical 2-subgroups of M_i using the modified local strategy and do the fusions in G as follows.

Suppose R is a radical p -subgroup of M . Using the local structure we can determine whether or not $N_M(R)$ is a subgroup of another maximal subgroup M' . Suppose $N_M(R)$ is a subgroup of M' . By Lemma 2.2, there is a radical subgroup R' of M' such that $R \leq R'$ and $N_M(R) \leq N_{M'}(R')$. Using local structure, we can determine whether or

not R is radical in M' and if so, we can identify R with a radical subgroup R' of M' . In this case, $N_M(R) =_G N_{M'}(R')$. Some more details are given below.

Case (2.1) We may take

$$\mathcal{R}_0(M_2, 2) = \{2^{1+12}, 2^{1+12}.2^3, 2^{1+12}.2^4, 2^{10}.2^{3+4}, 2^{3+12}.2^2, 2^{1+12}.2^{2+3}, 2^{1+12}.2^5, \\ 2^{1+12}.2^3.2^3, 2^{1+12}.2^2.2^4, 2^{1+12}.2.2^{2+3}, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2^{2+5}, 2^{1+12}.2.2^3.2^3, S\}$$

and by [20, Theorem 17], $N(R) = N_{M_2}(R)$ for all $R \in \mathcal{R}_0(M_2, 2)$, so that we may suppose $\mathcal{R}_0(M_2, 2) \subseteq \mathcal{R}_0(G, 2)$.

Let $R \in \mathcal{R}_0(M_2, 2) \setminus \{2^{1+12}\}$ and let $\sigma(R) : 1 < Q = 2^{1+12} < R$, so that $\sigma(R)' : 1 < R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.1. Thus there is a bijection g from $\mathcal{R}^-(\sigma(R), 2^{1+12})$ onto $\mathcal{R}^0(\sigma(R), 2^{1+12})$ such that $N(C') = N(g(C'))$ and $|C'| = |g(C')| - 1$ for each $C' \in \mathcal{R}^-(\sigma(R), 2^{1+12})$. So (6.2) holds and we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(M_2, 2) \setminus \{2^{1+12}\}} (\mathcal{R}^-(\sigma(R), 2^{1+12}) \cup \mathcal{R}^0(\sigma(R), 2^{1+12})).$$

In particular, $P_1 \notin_G \mathcal{R}_0(M_2, 2) \setminus \{2^{1+12}\}$ and if $P_1 = 2^{1+12}$, then $C =_G C(6)$. We may suppose

$$P_1 \in_G \{2^{10}, 2^{11}, 2^{10}.2^4, 2^{11}.2^4, 2^{3+12}, 2^{3+12}.2, 2^3.2^{6+8}, 2^2.2^{5+10}, \\ 2^{3+12}.D_8, 2^{3+12}.2^3, 2^{11}.2^{1+6}, 2^{3+12}.2^4, 2^{3+12}.2^3.2^2\}.$$

C		$N(C)$
$C(1)$	1	J_4
$C(2)$	$1 < 2^{11}$	$2^{11}: M_{24}$
$C(3)$	$1 < 2^{11} < 2^{1+12}.2^4$	$2^{1+12}.2^4.3.S_6$
$C(4)$	$1 < 2^{11} < 2^3.2^{6+8} < 2^{1+12}.2^2.2^4$	$2^{1+12}.2^2.2^4.(S_3 \times S_3)$
$C(5)$	$1 < 2^{11} < 2^3.2^{6+8}$	$2^3.2^{6+8}.(S_3 \times L_3(2))$
$C(6)$	$1 < 2^{1+12}$	$2^{1+12}.3.M_{22}.2$
$C(7)$	$1 < 2^{10} < 2^6.2^8$	$2^6.2^8.L_4(2)$
$C(8)$	$1 < 2^{10}$	$2^{10}: L_5(2)$
$C(9)$	$1 < 2^{10} < 2^{1+12}.2^3$	$2^{1+12}.2^3.(S_3 \times L_3(2))$
$C(10)$	$1 < 2^{10} < 2^{1+12}.2^3 < 2^{10}.2^{2+6}$	$2^{10}.2^{2+6}.(S_3 \times S_3)$
$C(11)$	$1 < 2^{10} < 2^{3+12}.2$	$2^{3+12}.2.(S_3 \times L_3(2))$
$C(12)$	$1 < 2^{10} < 2^{3+12}.2 < 2^6.2^8.2^3$	$2^6.2^8.2^3.L_3(2)$
$C(13)$	$1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3} < 2^{10}.2^2.2^{3+4}$	$2^{10}.2^2.2^{3+4}.S_3$
$C(14)$	$1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3}$	$2^{1+12}.2^{2+3}.(S_3 \times S_3)$
$C(15)$	$1 < 2^{3+12} < 2^{3+12}.2^2$	$2^{3+12}.2^2.(S_3 \times S_5)$
$C(16)$	$1 < 2^{3+12}$	$2^{3+12}.(S_5 \times L_3(2))$

Table 3: Some radical 2-chains of J_4

Case (2.2) Applying the local strategy [4] and [5], we get 16 radical subgroups of M_4 . Let

$$\mathcal{X} = \{2^{3+12}.2, 2^3.2^{6+8}, 2^2.2^{5+10}, 2^{3+12}.D_8, 2^{3+12}.2^3, 2^{3+12}.2^4, 2^{3+12}.2^3.2^2\},$$

so that each subgroup of \mathcal{X} is radical in G and contained in M_4 by [20]. Using the local structures, we can identify each $R \in \mathcal{X}$ with a radical subgroup of M_4 .

Next we consider fusions of subgroups in $\mathcal{R}_0(M_2, 2)$ and $\mathcal{R}_0(M_4, 2)$. Let $R = 2^{3+12}.2^2 \in \mathcal{R}_0(M_4, 2)$, so that $Z(R) = 2$ and $N_{M_4}(R) = 2^{3+12}.2^2.(S_3 \times S_5)$. Now G has two classes of involutions $2A$ and $2B$ such that $C(2A) = M_2$ and $C(2B) = 2^{11}.M_{22}:2$. Since $|N_{M_4}(R)|_2 > |C(2B)|_2$, it follows that $Z(R)$ is a $2A$ involution. In particular, we may suppose $N_{M_4}(R) \leq M_2$. By Lemma 2.2, there is a radical subgroup $P \in \mathcal{R}_0(M_2, 2)$ such that $R \leq P$ and $N_{M_4}(R) \leq N_{M_2}(P)$. By the local structures of subgroups of $\mathcal{R}_0(M_2, 2)$, R is a radical subgroup of M_2 such that $N_{M_4}(R) = N_{M_2}(R) = N(R)$, and so a radical subgroup of G .

Applying the local strategy to $N_{M_4}(R)$, we have that $N_{M_4}(R)$ has exactly 7 radical subgroups W and each is radical in M_4 with $N_{N_{M_4}(R)}(W) = N_{M_4}(W)$. In particular,

$$\mathcal{R}_0(N_{M_4}(R), 2) = \mathcal{R}_0(M_4, 2) \setminus (\mathcal{X} \cup \{2^{3+12}\}). \quad (6.3)$$

If we view R as a subgroup of $\mathcal{R}_0(M_2, 2)$ (we can identify R with a radical subgroup of M_2 using local structures), then each radical subgroup W of $N_{M_2}(R) = N_{M_4}(R)$ is radical in M_2 with $N_{M_2}(W) = N_{N_{M_2}(R)}(W)$. It follows that each $W \in \mathcal{R}_0(N_{M_4}(R), 2)$ is radical in G with $N(W) = N_{M_4}(W)$ and so each $Q \in \mathcal{R}_0(M_4, 2)$ is radical in G with $N(Q) = N_{M_4}(Q)$.

We may take

$$\mathcal{R}_0(M_4, 2) = \{2^{3+12}, 2^{3+12}.2, 2^{3+12}.2^2, 2^3.2^{6+8}, 2^2.2^{5+10}, 2^{3+12}.D_8, 2^{1+12}.2^5, 2^{1+12}.2^{2+3}, 2^{3+12}.2^3, 2^{1+12}.2^2.2^4, 2^{3+12}.2^4, 2^{1+12}.2^3.2^3, 2^{1+12}.2^{2+5}, 2^{11}.2^2.2^3.2^4, 2^{3+12}.2^3.2^2, S\}$$

and $N(R) = N_{M_4}(R)$ for all $R \in \mathcal{R}_0(M_4, 2)$, so that we may suppose $\mathcal{R}_0(M_4, 2) \subseteq \mathcal{R}_0(G, 2)$.

Let $R \in \mathcal{X}$ and $\sigma(R) : 1 < Q = 2^{3+12} < R$, so that $\sigma(R)' : 1 < R$. Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.1. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{X}} (\mathcal{R}^-(\sigma(R), 2^{3+12}) \cup \mathcal{R}^0(\sigma(R), 2^{3+12})).$$

In particular, $P_1 \notin_G \mathcal{X}$ and if $P_1 = 2^{3+12}$, then $P_2 \notin_G \mathcal{X}$.

Let $K = 2^{3+12}.2^2.(S_3 \times S_5)$. We may take

$$\begin{aligned} \mathcal{R}_0(K, 2) &= \{2^{3+12}.2^2, 2^{1+12}.2^5, 2^{1+12}.2^{2+3}, 2^{1+12}.2^2.2^4, \\ &\quad 2^{1+12}.2^3.2^3, 2^{1+12}.2^{2+5}, 2^{11}.2^2.2^3.2^4, S\} \subseteq \mathcal{R}_0(M_4, 2) \end{aligned}$$

and $N_K(R) = N_{M_4}(R) = N(R)$ for all $R \in \mathcal{R}_0(K, 2)$ and

$$\mathcal{R}_0(K, 2) = \mathcal{R}_0(M_4, 2) \setminus (\mathcal{X} \cup \{2^{3+12}\}).$$

Let $R \in \mathcal{R}_0(K, 2) \setminus \{2^{3+12}.2^2\}$ and let $\sigma(R) : 1 < 2^{3+12} < Q = 2^{3+12}.2^2 < R$, so that $\sigma(R)' : 1 < 2^{3+12} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{3+12}.2^2.(S_3 \times S_5), 2)} (\mathcal{R}^-(\sigma(R), 2^{3+12}.2^2) \cup \mathcal{R}^0(\sigma(R), 2^{3+12}.2^2)).$$

In particular, if $P_1 =_G 2^{3+12}$, then $C \in_G \{C(15), C(16)\}$.

Case (2.3) The fusions in G of subgroups of $\mathcal{R}_0(M_1, 2)$ with other subgroups in other $\mathcal{R}_0(M_i, 2)$ is similar to that of Case (2.2). We may take

$$\mathcal{R}_0(M_1, 2) = \{2^{11}, 2^{11}.2^4, 2^{1+12}.2^4, 2^3.2^{6+8}, 2^{11}.2^{1+6}, 2^{3+12}.D_8, 2^{1+12}.2^2.2^4, 2^{1+12}.2.2^{2+3}, 2^{3+12}.2^4, 2^{1+12}.2^{2+5}, 2^{3+12}.2^3.2^2, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2.2^3.2^3, S\}$$

and $N(R) = N_{M_1}(R)$ for all $R \in \mathcal{R}_0(M_1, 2)$, so we may suppose $\mathcal{R}_0(M_1, 2) \subseteq \mathcal{R}_0(G, 2)$.

For $R \in \{2^{11}.2^4, 2^{11}.2^{1+6}\} \subseteq \mathcal{R}_0(M_1, 2)$, let $\sigma(R) : 1 < Q = 2^{11} < R$, so that $\sigma(R)' : 1 < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \{2^{11}.2^4, 2^{11}.2^{1+6}\}} (\mathcal{R}^-(\sigma(R), 2^{11}) \cup \mathcal{R}^0(\sigma(R), 2^{11})).$$

In particular, $P_1 \neq_G 2^{11}.2^4$ or $2^{11}.2^{1+6}$, and if $P_1 = 2^{11}$, then $P_2 \neq_G 2^{11}.2^4$ and $2^{11}.2^{1+6}$.

Let $L = 2^{1+12}.2^4.3.S_6$. We may take

$$\mathcal{R}_0(L, 2) = \{2^{1+12}.2^4, 2^{1+12}.2^2.2^4, 2^{1+12}.2.2^{2+3}, 2^{1+12}.2^{2+5}, 2^{11}.2^2.2^3.2^4, 2^{1+12}.2.2^3.2^3, S\} \subseteq \mathcal{R}_0(M_1, 2)$$

and $N_L(R) = N_{M_1}(R)$ for all $R \in \mathcal{R}_0(L, 2)$.

Let $R \in \mathcal{R}_0(L, 2) \setminus \{2^{1+12}.2^4\}$ and let $\sigma(R) : 1 < 2^{11} < Q = 2^{1+12}.2^4 < R$, so that $\sigma(R)' : 1 < 2^{11} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{1+12}.2^4.3.S_6, 2)} (\mathcal{R}^-(\sigma(R), 2^{1+12}.2^4) \cup \mathcal{R}^0(\sigma(R), 2^{1+12}.2^4)).$$

In particular, if $P_1 = 2^{11}$, then $P_2 \notin_G \mathcal{R}_0(L, 2) \setminus \{2^{1+12}.2^4\}$ and if moreover, $P_2 =_G 2^{1+12}.2^4$, then $C =_G C(3)$.

Let $H = 2^3.2^{6+8}.(S_3 \times L_3(2))$. We may take

$$\mathcal{R}_0(H, 2) = \{2^3.2^{6+8}, 2^{3+12}.D_8, 2^{1+12}.2^2.2^4, 2^{3+12}.2^4, 2^{1+12}.2^{2+5}, 2^{3+12}.2^3.2^2, 2^{11}.2^2.2^3.2^4, S\} \subseteq \mathcal{R}_0(M_1, 2)$$

and $N_H(R) = N_{M_1}(R)$ for all $R \in \mathcal{R}_0(H, 2)$.

Let $\mathcal{Y} = \{2^{3+12}.D_8, 2^{3+12}.2^4, 2^{3+12}.2^3.2^2\} \subseteq \mathcal{R}_0(H, 2)$, $R \in \mathcal{Y}$, and let $\sigma(R) : 1 < 2^{11} < Q = 2^3.2^{6+8} < R$, so that $\sigma(R)' : 1 < 2^{11} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{Y}} (\mathcal{R}^-(\sigma(R), 2^3.2^{6+8}) \cup \mathcal{R}^0(\sigma(R), 2^3.2^{6+8})).$$

In particular, if $P_1 = 2^{11}$, then $P_2 \notin_G \mathcal{Y}$ and if moreover, $P_2 = 2^3 \cdot 2^{6+8}$, then $P_3 \notin \mathcal{Y}$.

We may take

$$\mathcal{R}_0(2^{1+12} \cdot 2^2 \cdot 2^4 \cdot (S_3 \times S_3), 2) = \mathcal{R}_0(H, 2) \setminus (\mathcal{Y} \cup \{2^3 \cdot 2^{6+8}\})$$

and moreover, $N_{2^{1+12} \cdot 2^2 \cdot 2^4 \cdot (S_3 \times S_3)}(R) = N_H(R) = N_{M_1}(R)$.

For each $R \in \mathcal{R}_0(2^{1+12} \cdot 2^2 \cdot 2^4 \cdot (S_3 \times S_3), 2) \setminus \{2^{1+12} \cdot 2^2 \cdot 2^4\}$, let $\sigma(R) : 1 < 2^{11} < 2^3 \cdot 2^{6+8} < Q = 2^{1+12} \cdot 2^2 \cdot 2^4 < R$, so that $\sigma(R)' : 1 < 2^{11} < 2^3 \cdot 2^{6+8} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{1+12} \cdot 2^2 \cdot 2^4 \cdot (S_3 \times S_3), 2) \setminus \{2^{1+12} \cdot 2^2 \cdot 2^4\}} (\mathcal{R}^-(\sigma(R), 2^{1+12} \cdot 2^2 \cdot 2^4) \cup \mathcal{R}^0(\sigma(R), 2^{1+12} \cdot 2^2 \cdot 2^4)).$$

It follows that if $P_1 = 2^{11}$, then $C =_G C(i)$ for some $2 \leq i \leq 5$.

Case (2.4) The fusions in G of subgroups of $\mathcal{R}_0(M_3, 2)$ with subgroups in other $\mathcal{R}_0(M_i, 2)$ is similar to that of Case (2.2). We may take

$$\mathcal{R}_0(M_3, 2) = \{2^{10}, 2^{10} \cdot 2^4, 2^6 \cdot 2^8, 2^{3+12} \cdot 2, 2^{1+12} \cdot 2^3, 2^{10} \cdot 2^{3+4}, 2^6 \cdot 2^8 \cdot 2^3, 2^{10} \cdot 2^{1+6}, 2^{1+12} \cdot 2^{2+3}, 2^{10} \cdot 2^{2+6}, 2^{3+12} \cdot 2^3, 2^{1+12} \cdot 2^3 \cdot 2^3, 2^{10} \cdot 2 \cdot 2^{3+5}, 2^{10} \cdot 2^2 \cdot 2^{3+4}, 2^{10} \cdot 2^6 \cdot 2^3, S\}$$

and $N(R) = N_{M_3}(R)$ for $R \in \mathcal{R}_0(M_3, 2) \setminus \mathcal{Z}$, where

$$\mathcal{Z} = \{2^6 \cdot 2^8, 2^6 \cdot 2^8 \cdot 2^3, 2^{10} \cdot 2^{1+6}, 2^{10} \cdot 2^{2+6}, 2^{10} \cdot 2 \cdot 2^{3+5}, 2^{10} \cdot 2^2 \cdot 2^{3+4}, 2^{10} \cdot 2^6 \cdot 2^3\}.$$

In addition, for $R \in \mathcal{Z}$, $C_{M_3}(2^6 \cdot 2^8) = 2^6$, $C_{M_3}(2^6 \cdot 2^8 \cdot 2^3) \simeq 2^3 \simeq C_{M_3}(2^{10} \cdot 2^{1+6})$,

$$C_{M_3}(2^{10} \cdot 2^{2+6}) \simeq C_{M_3}(2^{10} \cdot 2 \cdot 2^{3+5}) \simeq C_{M_3}(2^{10} \cdot 2^2 \cdot 2^{3+4}) \simeq 2,$$

$C_{M_3}(2^{10} \cdot 2^6 \cdot 2^3) \simeq 2^2$, and moreover,

$$N_{M_3}(R) = \begin{cases} 2^6 \cdot 2^8 \cdot L_4(2) & \text{if } R = 2^6 \cdot 2^8, \\ 2^6 \cdot 2^8 \cdot 2^3 \cdot L_3(2) & \text{if } R = 2^6 \cdot 2^8 \cdot 2^3, \\ 2^{10} \cdot 2^{1+6} \cdot L_3(2) & \text{if } R = 2^{10} \cdot 2^{1+6}, \\ 2^{10} \cdot 2^{2+6} \cdot (S_3 \times S_3) & \text{if } R = 2^{10} \cdot 2^{2+6}, \\ 2^{10} \cdot 2 \cdot 2^{3+5} \cdot S_3 & \text{if } R = 2^{10} \cdot 2 \cdot 2^{3+5}, \\ 2^{10} \cdot 2^2 \cdot 2^{3+4} \cdot S_3 & \text{if } R = 2^{10} \cdot 2^2 \cdot 2^{3+4}, \\ 2^{10} \cdot 2^6 \cdot 2^3 \cdot S_3 & \text{if } R = 2^{10} \cdot 2^6 \cdot 2^3. \end{cases}$$

Let $\sigma : 1 < Q = 2^{10} < 2^{10} \cdot 2^4$, so that $\sigma' : 1 < 2^{10} \cdot 2^4$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^{10}) \cup \mathcal{R}^0(\sigma, 2^{10})).$$

In particular, $P_1 \neq_G 2^{10} \cdot 2^4$ and if $P_1 = 2^{10}$, then $P_2 \neq_G 2^{10} \cdot 2^4$.

Let $J = 2^6 \cdot 2^8 \cdot L_4(2)$. We may take

$$\mathcal{R}_0(J, 2) = \{2^6 \cdot 2^8, 2^6 \cdot 2^8 \cdot 2^3, 2^{10} \cdot 2^{1+6}, 2^{10} \cdot 2^{2+6}, 2^{10} \cdot 2 \cdot 2^{3+5}, 2^{10} \cdot 2^2 \cdot 2^{3+4}, 2^{10} \cdot 2^6 \cdot 2^3, S\} \subseteq \mathcal{R}_0(M_3, 2)$$

and $N_J(R) = N_{M_3}(R)$ for all $R \in \mathcal{R}_0(J, 2)$.

Let $R \in \mathcal{R}_0(J, 2) \setminus \{2^6 \cdot 2^8\}$ and let $\sigma(R) : 1 < 2^{10} < Q = 2^6 \cdot 2^8 < R$, so that $\sigma(R)' : 1 < 2^{10} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^6 \cdot 2^8, L_4(2), 2)} (\mathcal{R}^-(\sigma(R), 2^6 \cdot 2^8) \cup \mathcal{R}^0(\sigma(R), 2^6 \cdot 2^8)).$$

In particular, if $P_1 = 2^{10}$, then $P_2 \notin_G \mathcal{R}_0(J, 2) \setminus \{2^6 \cdot 2^8\}$ and if moreover, $P_2 =_G 2^6 \cdot 2^8$, then $C =_G C(7)$.

Let $T = 2^{1+12} \cdot 2^3 \cdot (S_3 \times L_3(2))$. We may take

$$\begin{aligned} \mathcal{R}_0(T, 2) = & \{2^{1+12} \cdot 2^3, 2^{10} \cdot 2^{3+4}, 2^{1+12} \cdot 2^{2+3}, \\ & 2^{10} \cdot 2^{2+6}, 2^{1+12} \cdot 2^3 \cdot 2^3, 2^{10} \cdot 2 \cdot 2^{3+5}, 2^{10} \cdot 2^2 \cdot 2^{3+4}, S\} \subseteq \mathcal{R}_0(M_3, 2) \end{aligned}$$

and $N_T(R) = N_{M_3}(R)$ for all $R \in \mathcal{R}_0(T, 2)$.

Let $\mathcal{T} = \{2^{10} \cdot 2^{3+4}, 2^{1+12} \cdot 2^{2+3}, 2^{1+12} \cdot 2^3 \cdot 2^3\} \subseteq \mathcal{R}_0(T, 2)$, $R \in \mathcal{T}$, and let $\sigma(R) : 1 < 2^{10} < Q = 2^{1+12} \cdot 2^3 < R$, so that $\sigma(R)' : 1 < 2^{10} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{T}} (\mathcal{R}^-(\sigma(R), 2^{1+12} \cdot 2^3) \cup \mathcal{R}^0(\sigma(R), 2^{1+12} \cdot 2^3)).$$

In particular, if $P_1 = 2^{10}$, then $P_2 \notin_G \mathcal{T}$ and if moreover, $P_2 = 2^{1+12} \cdot 2^3$, then $P_3 \notin \mathcal{T}$.

Let $V = 2^{3+12} \cdot 2 \cdot (S_3 \times L_3(2))$. We may take

$$\begin{aligned} \mathcal{R}_0(V, 2) = & \{2^{3+12} \cdot 2, 2^6 \cdot 2^8 \cdot 2^3, 2^{1+12} \cdot 2^{2+3}, \\ & 2^{3+12} \cdot 2^3, 2^{1+12} \cdot 2^3 \cdot 2^3, 2^{10} \cdot 2^2 \cdot 2^{3+4}, 2^{10} \cdot 2^6 \cdot 2^3, S\} \subseteq \mathcal{R}_0(M_3, 2) \end{aligned}$$

and $N_V(R) = N_{M_3}(R)$ for all $R \in \mathcal{R}_0(V, 2)$.

Let $\sigma : 1 < 2^{10} < Q = 2^{3+12} \cdot 2 < 2^{3+12} \cdot 2^3$, so that $\sigma(R)' : 1 < 2^{10} < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^{3+12} \cdot 2) \cup \mathcal{R}^0(\sigma, 2^{3+12} \cdot 2)).$$

In particular, if $P_1 = 2^{10}$, then $P_2 \notin_G 2^{3+12} \cdot 2^3$ and if moreover, $P_2 = 2^{3+12} \cdot 2$, then $P_3 \neq 2^{3+12} \cdot 2^3$.

We may take

$$\mathcal{R}_0(2^{10} \cdot 2^{2+6} \cdot (S_3 \times S_3), 2) = \{2^{10} \cdot 2^{2+6}, 2^{10} \cdot 2 \cdot 2^{3+5}, 2^{10} \cdot 2^2 \cdot 2^{3+4}, S\} \subseteq \mathcal{R}_0(M_3, 2)$$

and moreover, $N_{2^{10} \cdot 2^{2+6} \cdot (S_3 \times S_3)}(R) = N_{M_3}(R)$.

For each $R \in \mathcal{R}_0(2^{10} \cdot 2^{2+6} \cdot (S_3 \times S_3), 2) \setminus \{2^{10} \cdot 2^{2+6}\}$, let $\sigma(R) : 1 < 2^{10} < 2^{1+12} \cdot 2^3 < Q = 2^{10} \cdot 2^{2+6} < R$, so that $\sigma(R)' : 1 < 2^{10} < 2^{1+12} \cdot 2^3 < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^{10} \cdot 2^{2+6} \cdot (S_3 \times S_3), 2) \setminus \{2^{10} \cdot 2^{2+6}\}} (\mathcal{R}^-(\sigma(R), 2^{10} \cdot 2^{2+6}) \cup \mathcal{R}^0(\sigma(R), 2^{10} \cdot 2^{2+6})).$$

It follows that if $P_1 = 2^{10}$ and $P_2 = 2^{1+12}.2^3$, then $C \in \{C(9), C(10)\}$.

We may take

$$\mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2) = \{2^6.2^8.2^3, 2^{10}.2^2.2^{3+4}, 2^{10}.2^6.2^3, S\} \subseteq \mathcal{R}_0(M_3, 2)$$

and moreover, $N_{2^6.2^8.2^3.L_3(2)}(R) = N_{M_3}(R)$.

For each $R \in \mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2) \setminus \{2^6.2^8.2^3\}$, let $\sigma(R) : 1 < 2^{10} < 2^{3+12}.2 < Q = 2^6.2^8.2^3 < R$, so that $\sigma(R)' : 1 < 2^{10} < 2^{3+12}.2 < R$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin \bigcup_{R \in \mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2) \setminus \{2^6.2^8.2^3\}} (\mathcal{R}^-(\sigma(R), 2^6.2^8.2^3) \cup \mathcal{R}^0(\sigma(R), 2^6.2^8.2^3)).$$

So if $P_1 = 2^{10}$ and $P_2 = 2^{3+12}.2$, then $P_3 \notin_G \mathcal{R}_0(2^6.2^8.2^3.L_3(2), 2) \setminus \{2^6.2^8.2^3\}$, and if moreover, $P_3 = 2^6.2^8.2^3$, then $C =_G C(11)$.

We may take

$$\mathcal{R}_0(2^{1+12}.2^{2+3}.(S_3 \times S_3), 2) = \{2^{1+12}.2^{2+3}, 2^{1+12}.2^3.2^3, 2^{10}.2^2.2^{3+4}, S\} \subseteq \mathcal{R}_0(M_3, 2)$$

and moreover, $N_{2^{1+12}.2^{2+3}.(S_3 \times S_3)}(R) = N_{M_3}(R)$.

Let $\sigma : 1 < 2^{10} < 2^{3+12}.2 < Q = 2^{1+12}.2^{2+3} < 2^{1+12}.2^3.2^3$, so that $\sigma' : 1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^3.2^3$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^{1+12}.2^{2+3}) \cup \mathcal{R}^0(\sigma, 2^{1+12}.2^{2+3})).$$

It follows that if $P_1 = 2^{10}$ and $P_2 = 2^{3+12}.2$, then $P_3 \neq_G 2^{1+12}.2^3.2^3$ and if moreover, $P_3 = 2^{1+12}.2^{2+3}$, then $P_4 \neq_G 2^{1+12}.2^3.2^3$.

Let $\sigma : 1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3} < Q = 2^{10}.2^2.2^{3+4} < S$, so that $\sigma' : 1 < 2^{10} < 2^{3+12}.2 < 2^{1+12}.2^{2+3} < S$. A similar proof to that of Case (2.1) shows that we may suppose

$$C \notin (\mathcal{R}^-(\sigma, 2^{10}.2^2.2^{3+4}) \cup \mathcal{R}^0(\sigma, 2^{10}.2^2.2^{3+4})).$$

It follows that if $P_1 = 2^{10}$, then $C \in \{C(i) : 7 \leq i \leq 14\}$. □

7 The proof of Dade's ordinary conjecture

Let $L = N(C)$ be the normalizer of a radical p -chain. If L is a maximal subgroup of J_4 , then the character table of L can be found in the library of character tables distributed with GAP. If L is not a maximal subgroup, its character table can be calculated easily using MAGMA.

The tables listing degrees of irreducible characters referenced in the proof of Theorem 7.1 are available electronically [6].

Theorem 7.1 *Let B be a p -block of $G = J_4$ with a positive defect. Then B satisfies Dade's ordinary conjecture.*

PROOF: By [13] and [11], we may suppose $p = 2$ or 3 and a defect group of B is non-cyclic. By Lemma 5.1, $B \in \{B_0, B_1, B_2\}$ when $p = 3$ and $B = B_0$ when $p = 2$.

Case (1) Suppose $p = 3$. If $B = B_2$, then

$$k(N(C(1)), B_2, d) = k(N(C(4)), B_2, d) = \begin{cases} 9 & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$k(N(C(3)), B_2, d) = k(N(C(2)), B_2, d) = \begin{cases} 18 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem when $B = B_2$.

If $B = B_0$ or B_1 , then

$$k(N(C(1)), B, d) = k(N(C(2)), B, d) = \begin{cases} 9 & \text{if } d = 3, \\ 5 & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$k(N(C(3)), B, d) = k(N(C(4)), B, d) = \begin{cases} 9 & \text{if } d = 3, \\ 2 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This proves the theorem when $p = 3$.

Case (2) Suppose $p = 2$, so that $B = B_0$. We set $k(i, d, u) = k(N(C(i)), B, d, u)$ for integers i, d, u .

We first consider the chains C with $d(N(C)) = 20$, so that $C =_G C(i)$ for $7 \leq i \leq 14$. The values $k(i, d, u)$ are given in Table 4.

Defect d	20	19	18	17	16	15	14	13	12	11	10	otherwise
$k(7, d, u)$	32	16	36	28	2	2	10	2	0	0	1	0
$k(8, d, u)$	32	16	20	8	2	0	2	0	0	0	1	0
$k(9, d, u)$	32	32	52	24	18	8	18	12	2	2	1	0
$k(10, d, u)$	32	32	84	100	26	26	26	14	6	4	1	0
$k(11, d, u)$	32	56	28	16	38	32	6	0	0	0	0	0
$k(12, d, u)$	32	56	44	36	38	34	14	2	0	0	0	0
$k(13, d, u)$	32	72	92	124	62	58	30	22	10	2	0	0
$k(14, d, u)$	32	72	60	48	54	40	22	20	6	0	0	0

Table 4: Values of $k(i, d, u)$ when $p = 2$ and $d(N(C(i))) = 20$

It follows that

$$\sum_{i=7}^{14} (-1)^{|C(i)|} k(N(C(i)), B_0, d) = 0.$$

Finally, suppose $C = C(i)$ is a chain with $d(N(C)) = 21$. Then $C =_G C(i)$ for $1 \leq i \leq 6$ or $15 \leq i \leq 16$. The values $k(i, d, u)$ are given in Table 5.

Defect d	21	20	19	18	17	16	15	14	13	12	11	7	otherwise
k(1, d, u)	32	8	4	4	1	0	4	4	1	0	0	1	0
k(2, d, u)	32	8	4	12	7	0	4	4	1	0	0	0	0
k(3, d, u)	32	24	12	28	11	12	11	12	11	6	1	0	0
k(4, d, u)	32	40	28	40	21	30	27	14	15	4	1	0	0
k(5, d, u)	32	24	20	24	9	18	20	6	1	0	0	0	0
k(6, d, u)	32	24	12	4	1	4	5	12	11	6	1	1	0
k(15, d, u)	32	40	28	16	11	22	21	14	15	4	1	0	0
k(16, d, u)	32	24	20	16	3	18	20	6	1	0	0	0	0

Table 5: Values of $k(i, d, u)$ when $p = 2$ and $d(N(C(i))) = 21$

It follows that

$$\sum_{d(N(C))=21} (-1)^{|C|} k(N(C), B_0, d, u) = 0$$

and Theorem 7.1 follows. □

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