MSM120—1M1
First year mathematics for civil engineers
Revision notes 1

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Introduction It is obvious that you can't do civil engineering (or any other kind of engineering) properly without a certain amount of mathematics. You will have done some of the mathematics you need at A-level, or the equivalent, but there is a lot more that will be useful to you.

If you are going to end up building bridges and such like things, then you are going to have to do a fair amount of mathematics to calculate stresses in the bridge components, to make sure the bridge is strong enough to cope with the traffic it is designed for. (This part of mechanics is called *Statics*.) At a more advanced level, you need to take account of the *Dynamics* of the structure as well. This involves solving differential equations to predict the way the bridge will move, or oscillate. You wouldn't want to make the mistakes made in the Millenium Bridge in London, which oscillates so much it has been declared unsafe.

In statics the basic tools are *trigonometry*, for resolving forces, surveying sites using triangulation, etc., and *integration* for calculating the cumulative effect of distributed loads, especially where complicated shapes are involved.

For example, suppose a bridge is made from a semi-circular arch of radius ℓ and width w, and thickness t at its thinnest point, supported by two pillars of depth d and height h. If the density of the material is ρ, what is the total weight of the bridge? (Draw a picture, and work out the answer for yourself. Then check that you get the answer $\rho w(2\ell(\ell + t) + 2hd - \frac{1}{2}\pi\ell^2)$.)

We begin the course with a review of material which most of you will be familiar with. We do this because it is essential that you are all completely on top of this material before we go on to more advanced topics. Then we discuss *differentiation* at some length, supported by the topic of *series*, especially *power series*.

The recommended textbook for this course is *Engineering Mathematics* by K. A. Stroud. If you are having difficulty with the lectures, you may find Stroud's explanations helpful. You are strongly encouraged to invest in a copy of this book.

There are certain basic topics which are no longer guaranteed to be in the A-level syllabus, but which we need. The first of these is:

Long division of polynomials For example, $x^4 - x^2 - x + 1$ has a root $x = 1$, so is divisible by $x - 1$. How do we find the quotient? The basic algorithm (i.e. method) is just like long division of integers, except that it is easier because there is no guesswork involved, and no carrying. At each step we divide the *leading term* of the divisor into the *leading term* of the remainder so far, to get the next term of the answer.

$$
\begin{array}{rrrrrr}
 & & x^3 & +x^2 & & -1 \\ \cline{2-6}
x-1 & \big) \; x^4 & & -x^2 & -x & +1 \\
 & x^4 & -x^3 & & & \\ \cline{2-3}
 & & x^3 & -x^2 & & \\
 & & x^3 & -x^2 & & \\ \cline{3-4}
 & & & & -x & +1 \\
 & & & & -x & +1 \\ \cline{5-6}
 & & & & & 0
\end{array}
$$

Try the following example yourself: divide $x^4 - 2x^3 + 4x$ by $x^2 + x + 1$. This time it does not go exactly: there is a *remainder* left at the end. Check that you have a quotient $x^2 - 3x + 2$ and remainder $5x - 2$. This means that

$$x^4 - 2x^3 + 4x = (x^2 + x + 1)(x^2 - 3x + 2) + (5x - 2)$$

or in other words

$$\frac{x^4 - 2x^3 + 4x}{x^2 + x + 1} = x^2 - 3x + 2 + \frac{5x - 2}{x^2 + x + 1}.$$

Sums of geometric series (See F.7 in Stroud)

Example: $S = 1 + 2 + 4 + 8 + 16 + 32$. This is a sum of *six* terms, where the *first term* is 1, and the *common ratio* is 2 (that is, each term is twice the previous term). If we multiply the equation through by 2 (i.e. the common ratio), we get

$$2S = 2 + 4 + 8 + 16 + 32 + 64.$$

Subtracting one equation from the other we get

$$
\begin{array}{rcccccccccccccc}
2S - S & = & & & 2 & + & 4 & + & 8 & + & 16 & + & 32 & + & 64 \\
 & & -(1 & + & 2 & + & 4 & + & 8 & + & 16 & + & 32) & & \\
\text{so} \quad S & = & -1 & & & & & + & & & & & & & 64 \\
 & = & 63 & & & & & & & & & & & &
\end{array}
$$

More generally, if we have a geometric series of n terms, where the first term is a and the common ratio is r, then the sum of the series is S, where

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

so that

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$$

and subtracting one equation from the other we obtain

$$(r-1)S = ar^n - a = a(r^n - 1)$$

since all the other terms cancel out. Therefore

$$S = a.\frac{r^n - 1}{r - 1} = a.\frac{1 - r^n}{1 - r}.$$

Another example: $S = 36 - 12 + 4 - \frac{4}{3} + \cdots$ to 7 terms. Here we have $a = 36 = 2^2.3^2$, and $r = -\frac{1}{3}$, and $n = 7$. Therefore

$$S = 36.\frac{1 - (\frac{1}{3})^7}{1 + \frac{1}{3}} = \frac{4.9.(1 + 3^{-7})}{4/3} = 3^3 + 3^{-4}.$$

Exponentials and logarithms If a is a real number and n is a positive integer (whole number) then you define $a^n = a.a.\cdots.a$, the product of n copies of a, so that $a^2 = a.a$ and $a^3 = a.a.a$ etc. Then it is easy to deduce the following laws of exponents:

$$\begin{aligned} a^m.a^n &= a^{m+n} \\ (a^m)^n &= a^{mn} \\ a^{-n} &= 1/a^n \\ a^m/a^n &= a^{m-n} \end{aligned}$$

Indeed, it is possible to generalise this to the case where n is *any real number*, and the same laws apply. For example $a^{1/2}.a^{1/2} = a^{1/2+1/2} = a^1 = a$, so $a^{1/2}$ is a square root of a.

Logarithms are defined as the "opposite" of raising a number to a power in this way. So if $a^x = y$, we say that x is the *logarithm* of y (to the base a). This is written $x = \log_a(y)$. If the base a is not specified, it should always be assumed to be $e \approx 2 \cdot 71828$. This special number is the *base of natural logarithms*, and is chosen because it simplifies lots of formulae which you will see later on. It really is a *natural* choice of base. Many people write $\ln x$ for $\log_e(x)$.

From the laws of exponents given above we can deduce corresponding laws of logarithms.

$$\begin{aligned} \log_a(xy) &= \log_a(x) + \log_a(y) \\ \log_a(x^y) &= y.\log_a(x) \\ \log_a(1/x) &= -\log_a(x) \\ \log_a(x/y) &= \log_a(x) - \log_a(y) \end{aligned}$$

Let us prove the first of these as an example:

Suppose that $p = \log_a(x)$ and $q = \log_a(y)$. Then $x = a^p$ and $y = a^q$, so $xy = a^p.a^q = a^{p+q}$, which means that $\log_a(xy) = p + q = \log_a(x) + \log_a(y)$, as required.

Radians Just as e is the "natural" base for logarithms, so π is the "natural" base for trigonometric functions. One whole revolution of a circle corresponds to an arc length equal to the whole circumference of the circle, that is $2\pi r$, where r is the radius. For simplicity, take a circle of radius 1, so that the circumference is 2π. So $360°$ corresponds to an arc length of 2π. Therefore an angle of $\theta°$ corresponds to an arc length of $2\pi.\frac{\theta}{360}$. This last figure is the same angle measured in *radians*. Thus for example, $\frac{\pi}{2}$ radians equals $90°$. To convert from degrees to radians, multiply by $\frac{\pi}{180}$, and to convert from radians to degrees, multiply by $\frac{180}{\pi}$.

Binomial expansions (See F.7 in Stroud)
Multiplying out we obtain $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$, and then

$$\begin{aligned}
(a+b)^3 &= (a+b)(a+b)^2 &&= (a+b)(a^2+2ab+b^2) \\
& &&= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= (a+b)(a+b)^3 &&= (a+b)(a^3+3a^2b+3ab^2+b^3) \\
& &&= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

Each coefficient in the right hand side here is obtained by adding together the two nearest coefficients in the row above: for example, the term in a^2b^2 in the last row above is obtained from a times $3ab^2$, plus b times $3a^2b$, giving a coefficient of $6 = 3+3$. Thus we can build up a triangle of these coefficients, and for each new entry, we just add together the two nearest entries in the row above. This is called Pascal's triangle (although it was well-known centuries before the time of Pascal).

$$\begin{array}{ccccccccccccc}
 & & & & 1 & & & & 1 & & & & \\
 & & & 1 & & & 2 & & & 1 & & & \\
 & & 1 & & & 3 & & 3 & & & 1 & & \\
 & 1 & & & 4 & & 6 & & 4 & & & 1 & \\
1 & & 5 & & 10 & & & & 10 & & 5 & & 1 \\
1 & 6 & & 15 & & & 20 & & & 15 & & 6 & 1
\end{array}$$

There is a formula for the entries in Pascal's triangle: the $(k+1)$th entry in the nth row is

$$\frac{n.(n-1).\cdots.(n-k+1)}{1.2.\cdots.k}$$

which can also be written as

$$\frac{n.(n-1).\cdots.(n-k+1)}{k!} \text{ or } \frac{n!}{(n-k)!k!},$$

where $n! = n.(n-1).(n-2).\ldots.3.2.1$ is n *factorial*, the product of all integers from 1 up to n.