Some Calculations Related To The Monster Group

by

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Synopsis

This thesis describes some computer calculations relating to the Monster group.

In Chapter 2 we describe the computer construction of the Monster carried out by Linton *et al.* in 1997, and again (over a different field) in 2000 by Wilson. We modify the latter programs, considerably increasing the speed and thus the feasibility of using them for practical computations.

We make use of our modified programs in Chapter 3 to determine conjugacy class representatives for the Monster group, up to algebraic conjugacy. This completes the work of Wilson *et al.* on conjugacy class representatives for sporadic simple groups.

Chapter 4 describes the method of "Fischer matrices" for computing character tables of groups with an elementary Abelian or extraspecial normal subgroup. Fischer's own rather cryptic paper on the subject has inspired many more expository papers by Moori *et al.* and our proof of Theorem 12 adds to this revised exposition. The final section of the chapter describes a GAP function to compute the character table of a group of shape $(2 \times 2{}^{\cdot}G){:}2$ and gives $(2 \times 2{}^{\cdot}\mathrm{Fi}_{22}){:}2 \leqslant \mathrm{Fi}_{24}$ as an example.

Chapter 5 finds most of the irreducible characters of the three-fold cover of a maximal subgroup of the Monster, $N(3B)$. We construct various representations which we first of all use to determine the conjugacy classes, power maps, and quotient map onto $N(3B)$. Observations in Chapter 3 aid us to find 463 of the 533 irreducible characters, but unfortunately we cannot find the remaining characters in the tensor algebra of our 463 irreducibles. As part of this work we also determine conjugacy class representatives for $6{}^{\cdot}\mathrm{Suz}$, including the algebraically conjugate classes.

Chapter 6 works towards a classification of the "nets" of the Monster. These are combinatorial structures defined by Norton to encode information about subgroups of the Monster in a way sympathetic with his generalised Moonshine conjectures. We witness all nets centralised by an element of prime order at least 5, and can determine which of these could be conjugate in the Monster. The remaining cases are considerably harder, and our programs will not scale up. Instead we describe our strategy to extend our work to a complete classification.

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Contents

List of Figures

Chapter 1

Introduction & Motivation

A group G with a normal subgroup N and quotient isomorphic to Q, *i.e.,* $G = N.Q$, may be understood through an investigation of Q and the action of Q on N. However, if G is simple then it has no proper non-trivial normal subgroups and so this reduction is not possible. We therefore find ourselves interested in simple groups, the building blocks of all groups. Let G be a finite simple group. According to the celebrated Classification Of Finite Simple Groups [14] G is one of the following groups:

(i) A cyclic group of prime order, C_p.

(ii) An alternating group A_n for $n \geqslant 5$.

(iii) A Chevalley group. For each of the root systems $X_n \in \{A_n, B_n, C_n, D_n, F_4, G_2, E_6, E_7, E_8\}$ one can obtain a finite group $X_n(q)$, most of which are simple. This approach provides a uniform construction of the classical groups, as well as a number of new infinite families.

- The classical groups are recovered as:
  - $\mathrm{A}_n(q) \cong \mathrm{L}_{n+1}(q)$ for $n \geqslant 1$.
  - ${}^2\mathrm{A}_n(q) \cong \mathrm{U}_{n+1}(q)$ for $n \geqslant 2$.
  - $\mathrm{B}_n(q) \cong \mathrm{O}_{2n+1}(q)$ for $n \geqslant 2$.
  - $\mathrm{C}_n(q) \cong \mathrm{S}_{2n}(q)$ for $n \geqslant 3$.
  - $\mathrm{D}_n(q) \cong \mathrm{O}^+_{2n}(q)$ for $n \geqslant 4$.
  - ${}^2\mathrm{D}_n(q) \cong \mathrm{O}^-_{2n}(q)$ for $n \geqslant 4$.
- The Suzuki groups $\mathrm{Sz}(q) = {}^2\mathrm{B}_2(2^{2m+1}) \cong {}^2\mathrm{C}_2(2^{2m+1})$ for $m \geqslant 2$.

- The Tits group ${}^2\mathrm{F}_4(2)'$.
- The characteristic 2 Ree groups, ${}^2\mathrm{F}_4(2^{2m+1})$ for $m \geqslant 2$.
- The characteristic 3 Ree groups, ${}^2\mathrm{G}_2(3^{2m+1})$ for $m \geqslant 2$.
- The other families are ${}^3\mathrm{D}_4(q)$, $\mathrm{G}_2(q)$, $\mathrm{F}_4(q)$, $\mathrm{E}_6(q)$, ${}^2\mathrm{E}_6(q)$, $\mathrm{E}_7(q)$, and $\mathrm{E}_8(q)$.

(iv) One of 26 other groups, the so-called "sporadic" simple groups. Extensive bibliographies on each of these groups are included in the Atlas of Finite Groups [8]. The smallest and longest known sporadic groups are the Mathieu groups M_i for $i \in \{11, 12, 22, 23, 24\}$. These were discovered in the 1860s and 1870s by Emil Mathieu.

The next sporadic simple group was not discovered until 1965. Janko discovered his group J_1 while classifying all finite simple groups with involution centraliser $2 \times \mathrm{L}_2(q)$ for odd q. The solution is ${}^2\mathrm{G}_2(q)$ for q an odd power of 3, or J_1 if $q = 5$.

Classifying groups with a specified involution centraliser also led to the discovery of Janko's other three groups, J_2, J_3, and J_4, the Held group He, the Lyons group Ly, and the Monster group $\mathbb{M}$.

When J_2 was constructed as a rank 3 primitive permutation group investigation of such groups revealed four more sporadic groups: The Higman-Sims group HS, the Suzuki group Suz, the McLaughlin group McL, and the Rudvalis group Ru.

Conway discovered his three sporadic groups Co_1, Co_2 and Co_3 by examining his perfect group Co_0, the automorphism group of the Leech lattice. He would have five groups to his name were it not for the previous discovery of HS and McL as these are also involved in Co_0.

Fischer's sporadic groups Fi_{22}, Fi_{23}, and Fi_{24}' were discovered as part of Fischer's classification of groups generated by a conjugacy class of involutions whose products have order at most 3. Increasing this order from 3 to 4 admits the Baby Monster sporadic group, B, and increasing to 6 admits the Monster.

The Thompson group Th, and Harada-Norton group HN, were discovered via their rôles as p-element centralisers in $\mathbb{M}$.

The Monster group is the largest of the 26 sporadic simple groups, having order

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71$$
$$= 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000$$

Its existence was predicted independently in 1973 by B. Fischer and R. Griess. In 1982 [15] Griess constructed the Monster as the automorphism group of his 196884-dimensional Griess algebra, thus proving existence. Uniqueness was proved in 1985 [36] by S. Norton.

The character table of the Monster was calculated in 1978 by B. Fischer, D. Livingstone, and M. Thorne [46]. The table was produced under the assumption that the smallest non-trivial irreducible has degree 196883. Usefully, Norton's uniqueness proof shows that any group in which the p-element centralisers are isomorphic to those of the Monster has a 196883-dimensional representation. The character degrees of the Monster are

$$1, 196883, 21296876, 842609326, \ldots$$

Extremely surprising is the co-incidence between the degrees of the irreducible characters of $\mathbb{M}$ and the coefficients in the Fourier expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots \qquad (q = \exp(2\pi i z))$$

of Klein's modular function. The coefficients are non-negative integer linear combinations of character degrees of $\mathbb{M}$. That all the coefficients are such combinations is part of the "Moonshine Conjectures" stated in 1982 by J. Conway and S. Norton [9]. Proof was provided in 1989 by R. Borcherds, and involves an infinite dimensional graded module for $\mathbb{M}$ where the representation on the ith homogeneous component is the representation suggested by the ith coefficient of the Fourier expansion of $j(\tau)$ [12].

In 1998 S. Norton made further conjectures [37] known as "generalised moonshine". This relates the character degrees of the p-element centralisers of the Monster to the coefficients of certain modular functions. One recovers the original moonshine conjectures for $p = 1$. To help understand the generalised moonshine phenomenon Norton defines "nets". These are polyhedra obtained by an action of the three string braid group on the cartesian product $X \times X \times X$ where X is the set of class $2A$ involutions in the Monster. This provides a link from $\mathbb{M}$ to the modular group.

The list of finite simple groups tells us little about the groups themselves. We should like to know properties such as what their conjugacy classes are, character tables, maximal subgroups, *etc.*. We may also like to have matrices or permutations that generate these groups available to us, so that we may perform calculations to further investigate a group, or as part of some other investigation.

The cyclic groups of prime order are easy to deal with. For the alternating groups the maximal subgroups are described by the O'Nan-Scott Theorem [28]. The conjugacy classes are easily deduced from those of the symmetric group, which correspond to partitions of n. It is easy to write down some generating permutations. The character table of A_n can be deduced from that of S_n, which can be computed [22].

The Chevalley groups are rather more complicated, but their organisation into infinite families allows the simultaneous investigation of many groups. The CHEVIE share package for GAP 3 [13] can compute conjugacy classes and the character table of a Chevalley group. The maximal subgroups are characterised by Aschbacher's theorem [4, 26].

There is little such similarity between any of the sporadic groups, and most must be investigated in isolation as a glance at the extensive bibliography in the ATLAS of Finite Groups [8] will reveal. Investigation of some of the sporadic groups is still ongoing: The maximal subgroups of $\mathbb{M}$ are not yet fully known [17, 19], and the 2-local analysis has only recently been completed [33].

A good deal of data on the simple groups is freely available as part of the GAP computing system. Furthermore, a selection of representations and useful computational data is available on the "Atlas of group representations" website [52].

We shall be interested in adding to this information, in particular information which is in some way related to the Monster group.

There are three computer constructions of the Monster group. The first [29] represents $\mathbb{M}$ as matrices over $\mathbb{F}_2$ in 196882 dimensions. This large dimension is an obvious obstruction to effective computation, and in general all we can do with these matrices is act on vectors. However, we can multiply together elements if they lie in a certain subgroup, $N(3B) = 3_+^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$. The second construction [50] is analogous to the first but in 196883 dimensions over $\mathbb{F}_7$. These are the two constructions we shall be using. The third construction of P. E. Holmes and R. A. Wilson [18] is on different generators and so is not of immediate use to us yet.

To start with we wish to finish the job R. A. Wilson *et al.* started [45, 51] of finding words for conjugacy class representatives in sporadic groups by doing so for the Monster. We shall employ the same strategy, *i.e.*, calculating the traces of matrices representing group elements. Of course we only have the 196882 and 196883 dimensional representations available to use, but these turn out to be sufficient to distinguish the conjugacy classes up to algebraic conjugacy. To work effectively with the 196883 dimensional representation over $\mathbb{F}_7$ we must modify the computer programs to increase the speed.

The main improvement is achieved by implementing "grease" for the operations in $N(3B)$. We also outline some experimental ideas, the most interesting is an implementation of arithmetic in $\mathbb{F}_7$ using logical and bitshift operations instead of a lookup table. This allows 8 field elements to be processed simultaneously instead of 2.

Finding conjugacy class representatives is a well documented and not terribly difficult procedure [45, 51]. Our situation is complicated by the difficulty of actually computing anything, and by having to search a large number of words. A database driven by Perl scripts allows us to keep track of what we have computed, and decide what to compute next.

We can produce representatives for classes $27A$ and $27B$ (which have the same character values and power maps) by working in $N(3B)$. Thus we obtain representatives for all conjugacy classes, up to algebraic conjugacy.

In the course of this work we became interested in the subgroup $N(3B) = 3_+^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$. It would have been useful to perform structure constant calculations using its character table, and use class fusion from it to the Monster. However, none of this information is known, and finding it is the subject of Chapter 5.

Our initial approach to this problem was to use "Fischer matrices". This turned out to be infeasible, but we did explain some of Fischer's rather cryptic paper [11], and produce a GAP program to construct the character table of a group of shape $(2 \times 2{\cdot}G){:}2$. This work constitutes Chapter 4.

The three-fold cover $3^{1+12}{:}6{\cdot}\mathrm{Suz}{:}2$ of $N(3B)$ is easier to work with as the extension is split and there is a low dimensional representation which exhibits the structure of the group nicely. We use this representation to determine the conjugacy classes, power maps, and the quotient map to $N(3B)$. Once we have the character table of $3^{1+12}{:}6{\cdot}\mathrm{Suz}{:}2$ the tables of $N(3B)$ and other quotients will follow.

The group $3^{1+12}{:}6{\cdot}\mathrm{Suz}{:}2$ has already been studied in some detail [29] and we are careful to be consistent with existing data. This allows us to use this data, as well as increasing the usefulness of our work and providing a check of correctness. We can also compute the class fusion from $N(3B)$ to $\mathbb{M}$ which was one of our original questions.

We use Clifford theory to help find the irreducible characters of $3^{1+12}{:}6{\cdot}\mathrm{Suz}{:}2$. We can compute some characters by constructing representations then lifting the eigenvalues of the matrices representing our class representatives to $\mathbb{C}$. These characters are computationally expensive to obtain, but tensoring known irreducibles with them and then reducing produces many more irreducibles.

Our consistency with existing data means that we can obtain reducible characters from a variety of sources. We can reduce these with our known irreducibles to find further irreducibles.

Let $G \leqslant \mathbb{M}$ be generated by three involutions in class $2A$ and be centralised by an element of class $3B$. Clearly $G \leqslant N(3B)$, so knowledge of $N(3B)$ will be useful if we wish to determine the conjugacy classes of such groups G. We shall need to do this in Chapter 6.

Specifically, Chapter 6 is concerned with "nets" in the Monster group. Hsu's exposition of "quilts" [20] (a net is a special kind of quilt) is useful for comparison, it highlights three problems:

(i) The connection between the quilt and properties of (in our case) the three string braid group.

(ii) Given a net obtained from involutions (a, b, c) what are the possibilities for the group $\langle a, b, c\rangle \leqslant \mathbb{M}$?

(iii) What are all the nets of the Monster?

Hsu has some partial solutions to the first two problems, we begin work on the third. That is, a complete classification of all nets, or equivalently all subgroups of $\mathbb{M}$ that are generated by a triple of $2A$ involutions. This is easy when the net—or rather the group generated by the three involutions—is centralised by an element of prime order $p \geqslant 7$ or an element in class $5B$: We work in the appropriate p-local maximal subgroup which presents no computational problems.

When a $5A$ element centralises the involutions we work in HN:2. This proves too difficult for our existing programs, but we overcome this by calculating "fingerprints" for the nets in order to distinguish them. A thorough fingerprinting algorithm produces good results. The remaining cases are even harder, and we end Chapter 6 by explaining some strategies to overcome the difficulties.

We witness all nets centralised by an element of prime order at least 5, and can determine which of these could be conjugate in the Monster.

Chapter 2

A Computer Construction Of The Monster

Browsing through the many representations available in the Web Atlas [52] one may suppose that for most groups there exist faithful representations in sufficiently small dimension to make computing with a matrix representation feasible. In fact this is not the case, not least of course because there are many more groups than those in [52]. The smallest faithful representation of the Monster group is in 196882 dimensions over $\mathbb{F}_2$ which makes a matrix construction infeasible—it simply takes too long to multiply such matrices together to be of any particular use.

In this chapter we describe the computer constructions of the Monster that we shall use for our calculations [29, 50].

(2.1) Computer Construction Of Groups

The method generally used to construct matrix representation of a finite group over a finite field is described in [43] and in Section 2.1.2 below.

(2.1.1) The Standard Basis Algorithm

"Standard Basis" is one of the programs of the Meat-Axe [42], usually called `zsb`. A standard basis is one which can be specified "independently of basis": Let G be a group generated by elements g and h. Let ρ be an irreducible matrix representation of G and write $A = \rho(g)$ and $B = \rho(h)$. Now let τ be another matrix representation of G in the same dimension and over the same field and write $C = \tau(g)$ and $D = \tau(h)$. Are these representations equivalent *i.e.,* is there a base change matrix P such that $A = P^{-1}CP$ and $B = P^{-1}DP$? A common basis is needed, and this is exactly the purpose of Standard Basis.

Consider the algebra generated by A and B, then some combination (under addition and multiplication), say $f(A, B)$, of A and B will have non-zero nullity (preferably nullity 1), so let $\mathbf{v}$ be the (uniqueness is of course only up to scalar multiplication) null vector. The standard basis algorithm now produces a basis as follows: Let $\mathbf{v}_1 = \mathbf{v}$ and suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ have already been chosen, then $\mathbf{v}_{n+1}$ is the first element of the sequence

$$\mathbf{v}_1,\ \mathbf{v}_1 A,\ \mathbf{v}_1 B,\ \mathbf{v}_2 A, \mathbf{v}_2 B,\ \ldots,\ \mathbf{v}_n A,\ \mathbf{v}_n B$$

that is linearly independent from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. If no such vector exists then the algorithm finishes and the output is a matrix P with rows the $\mathbf{v}_i$. Since ρ is irreducible the single vector $\mathbf{v}_1$ must "spin up" to span the whole space.

In the algebra generated by C and D consider the matrix $f(C, D)$ where f is as above. If this matrix has different nullity then the representations cannot be equivalent since conjugation preserves rank. If the nullity is 1 in both cases then run the standard basis algorithm using the null vector $\mathbf{w}$ of $f(C, D)$ and the generators C and D to obtain a base change matrix, T say.

The matrix $P = T^{-1}S$ is the required base change matrix for which $A = P^{-1}CP$ and $B = P^{-1}DP$. However, if these equations do not hold then the representations ρ and τ are inequivalent.

If the nullity is more than 1 then one may choose any null vector of $f(A, B)$. Each null vector of $f(C, D)$ must be taken in turn until either a matrix P such that $A = P^{-1}CP$ and $B = P^{-1}DP$ is found, or all possibilities are exhausted in which case the representations are inequivalent.

If the representations are equivalent and we have chosen corresponding null vectors then we obtain a matrix R such that $A^R = C$, $B^R = D$, and $\mathbf{v}R = \mathbf{w}$. Clearly $\mathbf{v}_i R = \mathbf{w}_i$ and so A written with respect to the standard basis $\{\mathbf{v}_i\}$ must be equal to C written to the standard basis $\{\mathbf{w}_i\}$. Likewise for B and D. Furthermore, this will be true for any matrix P that conjugates (A, B) to (C, D), though it does not follow that $P = R$. However, if ρ (and therefore τ) is irreducible then Schur's Lemma means that $P = \xi R$ for a non-zero scalar ξ.

(2.1.2) Subgroups H and K that intersect in a group L

Let G be a group generated by subgroups H and K that intersect in a group L. To construct a representation ρ of G, first of all construct a representation σ of H which is equivalent to $\rho{\downarrow}H$, and a representation τ of K which is equivalent to $\rho{\downarrow}K$. Now in H choose generators for L, and in K choose the same generators for L. The standard basis algorithm can now be used to change the basis of τ to get an equivalent representation τ' such that $\tau'{\downarrow}L = \sigma{\downarrow}L$. Note that there may be more than 1 representation τ' equivalent to τ and equal to σ on L, and all of these possibilities must be examined in order to obtain a group isomorphic to G generated by $H\sigma$ and $K\tau'$.

(2.2) Constructing The Monster Group

In the case of the Monster group, choose $H = 3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ and $K = 3^{2+5+10}{\cdot}(\mathrm{M}_{11} \times 2{\cdot}\mathrm{S}_4)$ intersecting in $3^{2+5+10}{\cdot}(\mathrm{M}_{11} \times \mathrm{S}_3)$.

Suppose that $g \in \mathbb{M}$ powers up to class $3B$ then $\langle g \rangle \leqslant N(3B)$ (for a chosen $3B$ element). Therefore if an element of $\mathbb{M}$ powers up into class $3B$ then its conjugacy class (and the classes of all its powers) intersect $N(3B)$ non-trivially. This observation will be used frequently. Note that it is a sufficient condition but not a necessary condition for membership of $N(3B)$.

(2.2.1) Representing $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}$

Let V be the 196882-dimensional $\mathbb{F}_2$-module for $\mathbb{M}$, and let χ be its character, so that $\chi = \chi_2 - \chi_1$ mod 2 where χ_2 and χ_1 are tabulated in the ATLAS [8]. Chapter 10 of [48] decomposes V restricted to $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}$, and this may be embellished as follows.

Let x be the central element of order 3, which is in class $3B$ of $\mathbb{M}$. Then $\left\langle \chi_2{\downarrow}\langle x \rangle, 1_{\langle x \rangle} \right\rangle = 65663$. By diagonalising the matrix of x it is clear that x has eigenvalue 1 with multiplicity 65663. Furthermore, since $\chi_2(x) \in \mathbb{Z}$ the other eigenvalues ω and $\overline{\omega}$ (where $\omega = \exp \frac{2\pi i}{3}$) must occur with equal multiplicity, namely 65610, corresponding to a space of dimension 131220 on which x acts faithfully. Reducing modulo 2 we see that the 196882-dimensional module must split as

$$\mathbf{196882} = \mathbf{65610} \oplus \mathbf{65610} \oplus \mathbf{65662}$$

and since x is central this splitting is respected by the whole of $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}$. The involution which extends this group to $N(3B)$ inverts x and so swaps the first two summands here.

The 65610-dimensional modules are for $3^{1+12}{\cdot}2{:}\mathrm{Suz}$ and have the structure $\mathbf{90} \otimes \mathbf{729}$ where $\mathbf{90}$ is a module for $6{\cdot}\mathrm{Suz}$ and $\mathbf{729}$ is a faithful module for $3^{1+12}{:}2{\cdot}\mathrm{Suz}{:}2$. Now, 3^{1+12} has 3^{12} linear characters lifted from 3^{12} and two 729-dimensional faithful characters which are dual to each other. Since the linear characters cannot be faithful on the central 3, each **65610** must contain only one of the two corresponding 729-dimensional representations. The 729-dimensional representations extend to $3^{1+12}{:}2{}^{\cdot}\mathrm{Suz}$ so it follows from Clifford theory (see Section 4.1) that the 65610-dimensional module has structure $\mathbf{90} \otimes \mathbf{729}$ where **90** is a module for $6{}^{\cdot}\mathrm{Suz}$.

The 65662-dimensional $\mathbb{C}$-module W on which x is represented trivially can be examined as follows. Take another element $y \in 3B$ such that $\langle x, y\rangle \cong C_3 \times C_3 \leqslant 3^{1+12}$ (this is denoted $3B^2$) then every non-trivial element in this group has character value 53 in χ_2. Hence

$$\left\langle 1_{\langle x,y\rangle}, \chi_2{\downarrow}\langle x, y\rangle \right\rangle = \frac{196883 + 8.53}{9} = 21923$$

is the dimension of its fixed space. Therefore $C_W(3B^2) < C_W(3B)$ and since $3B^2 \leqslant 3^{1+12}$ the action of 3^{1+12} on $C_W(3B)$ must be non-trivial. Since x is acting trivially, this corresponds to a non-trivial action of 3^{12}.

From [49] the group 3^{12} may be thought of as congruence classes of the complex Leech lattice, $\Lambda_{\mathbb{C}}$, modulo $\theta\Lambda_{\mathbb{C}}$ (where $\theta = \sqrt{-3}$). The action of ω (which corresponds to the central 3) is trivial and so this gives a representation of $2{}^{\cdot}\mathrm{Suz}$ on 3^{12}. The orbits have sizes 1, 65520, and 495920. From the fixed space argument above it is clear that the orbit of size 65520 must occur, and so

$$\mathbf{65662} = \mathbf{65520} \oplus \mathbf{142}$$

where **142** is a direct sum of irreducible $\mathbb{F}_2$-modules for $2{}^{\cdot}\mathrm{Suz}$. To show that **142** is irreducible it is sufficient to show that it is not trivial, since Suz has no non-trivial $\mathbb{F}_2$-irreducibles of smaller degree. Consider the action of an element $g \in 3^{1+12}{\cdot}2{}^{\cdot}\mathrm{Suz}$ of order 13 on the 196882-dimensional space. Find g as the cube of an element of order 39 so that g is in $\mathbb{M}$-class $13B$ and $\chi_2(g) = -2$ so $\chi(g) = 1$. By explicit computation one finds that the trace on each of the 65610-dimensional modules is -1. The trace on the 65520-dimensional module must be 0. This is because the 65520-dimensional representation is induced from the subgroup $3^{12}{:}\mathrm{U}_5(2)$ which contains no elements of order 13. The trace so far in characteristic 0 is -2, or 0 in characteristic 2. But $\chi(g) = 1$ so the trace of g on **142** must be 1, and thus **142** cannot be a direct sum of 142 one-dimensional modules.

There is a correspondence between $\mathbb{F}_2^2$ and $\mathbb{F}_4$, say:

$\mathbb{F}_2^2$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$\mathbb{F}_4$	0	1	ω	$\overline{\omega}$

which induces an isomorphism $\mathrm{GL}_2(2) \cong \Gamma L_1(4)$ given by

$\mathrm{GL}_2(2)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\Gamma \mathrm{L}_1(4)$	1	ω	$\overline{\omega}$	1^*	ω^*	$\overline{\omega}^*$

where $*$ denotes the action is semi-linear and not linear.

This is of great use in simplifying the construction of the 65520-dimensional module for 2·Suz.

The complex Leech lattice can be reduced modulo θ, as in [49], so that the group 2·Suz acts with an orbit of size 2×32760 on the congruence classes of type 2 vectors, each of which contains 3 vectors. This is now interpreted as an action on the 32760 pairs $\{x, -x\}$ where the central involution swaps the elements of the pair and Suz acts as permutations on 32760 points. Now, $\Lambda_{\mathbb{C}}/\theta \cong 3^{12}$ and lifting to the triple cover 3^{1+12} the congruence class x together with the new central element z generate a group of shape 3^2, the non-trivial proper subgroups of which can be identified with $\mathbb{F}_2^2$ via

$$\langle z \rangle \mapsto \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \qquad \langle x \rangle \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \qquad \langle zx \rangle \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \qquad \langle z^2 x \rangle = \langle z x^2 \rangle \mapsto \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \tag{1}$$

say. The action by conjugation of $y \in 3^{12}$ on $xz \in 3^{1+12}$ is given by

$$(xz)^y = x^y z = xz^k$$

for some $k \in \{0, 1, 2\}$, so the action of y can be realised as an element of $\mathrm{GL}_2(2)$ acting on the space $\mathbb{F}_2^2$ described in (1). Such elements clearly have order 1 or 3 and so the action of the central element of 2·Suz can be encoded as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Hence the action of 3^{12}:2·Suz can be realised as a permutation on 32760 points and a list of 32760 elements of $\mathrm{GL}_2(2)$: The action on a vector is obtained by treating a length 65520 vector over $\mathbb{F}_2$ as a length 32760 vector over $\mathbb{F}_4$.

A similar trick can be done with the 131220 dimensional representation of 3^{1+12}:2·Suz:2. First of all a 729 dimensional module for 3^{1+12}:2·Suz is constructed over $\mathbb{F}_4$. Applying the field automorphism of $\mathbb{F}_4$ gives another representation but with the central 3 now represented by $\overline{\omega}$ (say) instead of ω. Similarly for a 90 dimensional module for 6·Suz. Tensoring the correct 729 dimensional module with the correct 90 dimensional module gives the required representation for 3^{1+12}·2·Suz. Another representation for this group is obtained by tensoring the dual modules. Since the action of the outer automorphism is to invert the central 3, this is equivalent to simply swapping these two tensor products over. Now, by changing basis these representations can be written over $\mathbb{F}_2$ in the same dimension and so we obtain a module of shape $(\mathbf{90} \otimes \mathbf{729}) \oplus (\overline{\mathbf{90}} \otimes \overline{\mathbf{729}})$, say, with the outer automorphism swapping the summands. However, one summand is obtained from the other by applying the field automorphism and so it is sufficient to work with just one of these modules. Note that a representation for a non-isomorphic group also of shape $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ could be obtained using the module $(\overline{\mathbf{90}} \otimes \mathbf{729}) \oplus (\mathbf{90} \otimes \overline{\mathbf{729}})$.

The action of the tensor product is realised by converting the length 131220 vector into a length 65610 vector over $\mathbb{F}_4$ then folding it into a 90×729 matrix. This is then pre-multiplied by a 90×90 matrix (the transpose of the matrix for the action of $6{\cdot}\mathrm{Suz}$) and post-multiplied by a 729×729 matrix representing the action of $3^{1+12}{:}2{\cdot}\mathrm{Suz}$. The outer automorphism (if required) is obtained by applying the field automorphism of $\mathbb{F}_4$ to the output and swapping the $\mathbf{90} \otimes \mathbf{729}$ component with the $\overline{\mathbf{90}} \otimes \overline{\mathbf{729}}$ component.

(2.2.2) Another Construction Over $\mathbb{F}_7$

The construction over $\mathbb{F}_7$ [50] in which we are primarily interested closely follows the construction over $\mathbb{F}_2$. However, since $\mathbb{F}_7$ contains elements of order 3 the above translation between different fields is avoided. Over $\mathbb{F}_7$ the construction consists of two $\mathbf{90} \otimes \mathbf{729}$ modules that are swapped over by the outer automorphism, a monomial action on 65520 points, and a 143 dimensional representation of Suz.

(2.2.3) The Generators

The generators used for the subgroup $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ we have been describing are named A, B, C, D, and E. Generators C and D are standard generators for the subgroup $6{\cdot}\mathrm{Suz}{:}2$, and A and B are standard generators for $6{\cdot}\mathrm{Suz}{:}2$, so $B = D$.

The generator E is a specified element of 3^{1+12}. Specifically, E is inverted by the central involution of $6{\cdot}\mathrm{Suz}$ and is centralised by a certain subgroup $\mathrm{U}_5(2)$, standard generators for which are obtained from A and B by using the words on the web Atlas.

We know how these generators act on $\mathbb{F}_2^{196882}$ and $\mathbb{F}_7^{196883}$ and in the next section we shall describe how this is exploited to perform calculations.

A final generator, T, normalises $L_1 = 3^{2+5+10}{:}(\mathrm{M}_{11} \times 2^2)$ and extends $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ to the Monster. The action of T on L_1 can be described by specifying two sets of generators for L_1 which are swapped by T. This is the subject of Section 5.1 of [29], and Section 6 examines the action of T on each of the constituents of $V{\downarrow}L_1$, thus describing the action of T on 196882-dimensional $\mathbb{F}_2$-space.

(2.3) Programming The Construction

For speed the computer implementation of the construction is written in C. In particular, with the mod 2 construction we can store 8 field elements in a byte and perform arithmetic on 32-bit words *i.e.*, 32 field operations at once. Using a low level compiled language is also inherently faster than using a high level interpreted language such as GAP or Magma.

Programs can be found on the CD within `/monster`.

(2.3.1) Monster Operations (mop)

The $\mathbb{F}_2$ version of the "Monster operations" functions, contained in the file `/monster/mop2/mop.h`, were written by R. A. Parker and R. A. Wilson. An element of $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}{:}2$ is encoded as a `suzel` which contains:

- A 90×90 matrix over $\mathbb{F}_4$.
- A 729×729 matrix over $\mathbb{F}_4$.
- A flag, `inout`, which indicates whether the outer automorphism is applied.
- A permutation on 32760 points.
- A list of 32760 elements of $\mathrm{GL}_2(2)$ encoded as integers 1 to 6.
- A 142×142 matrix.

The action of a `suzel` on a vector is described in at the end of Section 2.2.1 and is implemented in the function `vecsuz`.

There are three such generators, named B, C, and E. The generators B and C are standard generators of the subgroup $6{\cdot}\mathrm{Suz}{:}2$ that complements 3^{1+12}. The third generator E is a specified element of 3^{1+12}. We can easily multiply these generators together, this is the purpose of the function `suzmult`. For example, if $Z = BC$ then the 90×90 matrix of Z is equal to that of C multiplied by that of D. Likewise with the

729×729 matrices, the permutations π_B and π_C, and the 142-dimensional matrices. Let $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{z}$ be the lists of elements of $\mathrm{GL}_2(2)$ of B, C, and Z, respectively. Then $\mathbf{z}_i = \mathbf{b}_i \mathbf{c}_{i\pi_B}$.

The other generator T is stored as a permutation on 87480 points, 87480 elements of S_3, two 324-dimensional matrices, and a 538-dimensional matrix. The submodule structure used to express T in this abbreviated form is of course different to the structure used to encode the other generators. It is therefore impossible to multiply T with any of the other generators

A new version of the functions, `mop7.h`, was made by R. A. Wilson [50] for the $\mathbb{F}_7$ construction. We modified this as described in the next section. In this version a `suzel` consists of:

- Two 90×90 matrices over $\mathbb{F}_7$, M_1 and M_2 say.
- Two 729×729 matrices over $\mathbb{F}_7$, L_1 and L_2 say.
- A flag `inout` which indicates whether the outer automorphism is applied.
- A permutation on 65520 points.
- 65520 elements of $\mathbb{F}_7$.
- A 143×143 matrix.

The two 90×90 matrices are stored as a single 180×90 matrix, and similarly L_1 and L_2 are stored as a single 1458×729 matrix. The `/monster/mop7/mop7.h` function `vecsuz` now works as follows. The input vector is split into segments of length 131220, 65520, and 143. The length 131220 segment is folded up into two 90×729 matrices, X_1 and X_2 say, stored as a single 180×729 matrix. If the `suzel` is inner then the output vector is the concatenation of $M_1(X_1L_1)$ and $M_2(X_2L_2)$, parentheses indicating the order in which we perform the calculation. If the `suzel` is outer then the output vector is the concatenation of $M_1(X_2L_2)$ and $M_2(X_1L_1)$. The length 65520 and 143 segments are acted on in the obvious ways.

In this case the generators are named C, D, and E. As usual C and D are standard generators for $6^{\cdot}\mathrm{Suz}{:}2$, and as in the $\mathbb{F}_2$ case we can freely multiply them together, but not with the other generator T.

The inability to multiply together T with any of the other generators is of course a handicap, but we can work round this. Let WT be a word in the generators where W is a word in $\langle B, C, E\rangle$. We know the action of W and T on the underlying space, so we can calculate the action of WT on a vector. We can guess the element order as the orbit length of a random vector. This can be improved to an exact order calculation by using two vectors whose stabilisers are known to intersect trivially, as described at the end of [29]. We can also calculate the trace of such an element by acting on each co-ordinate vector in turn and adding the appropriate entry of the result to a running total.

(2.3.2) Grease

The "mop" and "mop7" programs described in Section 2.3.1 were used for finding the orders of elements and traces.

Our modifications to the modulo 7 programs involved re-writing parts of them to include "grease", a matrix multiplication method which can produce considerable speed increases. (The modulo 2 programs are already greased.)

Let $A = (a_{ij})$ and $B = (b_{jk})$ be matrices over a finite field and which are to be multiplied, so $(AB)_{ik} = \sum_j a_{ij}b_{jk}$. Let $\mathbf{b}_j$ be the jth row of B and $\mathbf{c}_i$ be the ith row of AB. Then $\mathbf{c}_i = \sum_j a_{ij}\mathbf{b}_j$ and this is how we think of matrix multiplication when programming the computer. However, a_{ij} can take only a few different values—7 in this case. Therefore if the range of i is sufficiently large then for each j all 7 multiples of $\mathbf{b}_j$ are likely to be used more than once. The program is said to be greased to level 1 when all 7 multiples of all $\mathbf{b}_j$ are calculated in advance of the matrix multiplication, and simply looked up when they are needed.

In the case of the computer construction of the Monster A has size 729×729 and B is 729×90 and so it is efficient to grease to level 2: Calculate in advance of the matrix multiplication all linear combinations $\lambda\mathbf{b}_j + \mu\mathbf{b}_{j+1}$ for odd j.

We re-wrote parts of `mop7.h` to include level 2 grease. For multiplying `suzel`s we greased the 729×729 matrices and the 90×90 matrices. We also grease the 90×729 folded vectors too. The changes to `mop7.h` are described briefly in Appendix A.1, and in the file `/monster/mop7/mop7.h`.

(2.3.3) Driver Programs

We also wrote a number of driver programs to use `mop.h` and `mop7.h` that we shall make use of in Chapter 3. The most important are:

`/monster/mop2/melts.c`

Usage: `% ./melts infile outfile`

Description: Calculate the orders of the Monster elements corresponding to the words contained in the input file.

Details: On each line of the input file, `infile`, is a word W in the generators B and C. The program calculates the order of the Monster element corresponding to the word WT and writes to the output file the word WT, on the next line the order of the Monster element, then a blank line to delimit entries.

`/monster/mop2/mels.c`

Usage: `% ./mels infile outfile`

Details: As `melts.c` but calculates the order of the Monster element corresponding to the word W rather than of WT. Moreover, W may be a word in B, C, and E.

`/monster/mop2/char.c`

Usage: `% ./char infile outfile errfile`

Description: Calculate the traces of the Monster elements corresponding to the words contained in the input file.

Details: A record in the input file consists of 3 lines and records are delimited by an empty line. The first line contains a word WT for a Monster element g where W is a word in B and C. The second line contains an integer n, the program will calculate the trace of g^n. The third line is a comment. In practise we put the order of g (as calculated by `melts.c`) on this line and check that `char.c` agrees.

A record in the output file consists of 5 lines: The word WT, the order of g, the comment line from the input file, n, and the trace of g^n.

When the trace calculations for a word WT begin the file `errfile` is initialised and used to log WT and the partial trace every 10 000 co-ordinates. If this file exists when the program is launched then it reads WT and the latest partial trace from `errfile`. The program then scans the input file for WT and resumes calculations.

`/monster/mop2/class2.c`

Usage: `% ./class2 infile outfile`

Description: The input file contains straight-line programs in C, D, and E which will be evaluated in the Monster. For each Monster element g the program calculates the trace modulo 2 of g and of certain powers of g so that, together with the correct modulo 7 information about g, the conjugacy class of g can be determined using the information in Appendix B.

Details: The input file consists of straight-line programs for Monster elements. Each line contains one of the following 4 commands:

`w i` Begin a new element, the ith in the input file.

`p i j k` $g_k := g_i^j$.

`m i j k` $g_k := g_i \times g_j$.

`r i` g_i is the element returned by this straight-line program.

Having made an element g the program calculates the order of g and uses the conjugacy class identification data in Appendix B (specially formatted) in a file `cid_data_c.txt` to determine those n such that it should calculate the trace of g^n.

Line i in the output file corresponds to the ith straight-line program in the input file. The output line is of the form

`o`n $n_1.t_1$ $n_2.t_2 \ldots$

where g has order n and the trace of g^{n_j} is t_j.

To determine the conjugacy class of g it is generally necessary to have corresponding output from the mod 7 version of this program, `monster/mop7/class7.c`, see below. A Perl script processes both of the output files and uses the information in Appendix B to determine the conjugacy classes of the elements.

`/monster/mop7/char.c`

Usage: `% ./char infile outfile errfile`

Description: Calculate the traces modulo 7 of the Monster elements corresponding to the words contained in the input file.

Details: Modulo 7 version of `/monster/mop2/char.c`.

`/monster/mop7/class7.c`

Usage: `% ./class7 infile outfile`

Description: The input file contains straight-line programs in C, D, and E which will be evaluated in the Monster. For each Monster element g the program calculates the trace modulo 7 of g and of certain powers of g so that, together with the correct modulo 2 information about g, the conjugacy class of g can be determined using Appendix B.

Details: Modulo 7 version of `/monster/mop2/class2.c`.

(2.3.4) Other Modifications

In `/monster/mop2/mop2.h` we removed a number of global variables, in particular ones used for temporary storage by various functions. This allowed us to reliably implement dynamic allocation (and de-allocation) of memory for `suzel`s, *i.e.,* it is no-longer necessary to explicitly declare and allocate memory for every `suzel` at the start of a program.

The programs that implement the $\mathbb{F}_2$ construction of [29] are based on R. Parker's C MeatAxe which stores an element of $\mathbb{F}_2$ as a single bit so that addition and multiplication can be done 32 elements at once (on a 32-bit computer) using the operations `xor` and `and` (respectively). With this in mind we modified the modulo 7 programs as follows. Originally each byte stored two field elements, say (a, b), as $7a + b$ and arithmetic was done with a 256×256 byte lookup table, two field elements at a time. In our modified packing we store a in the first nibble of the byte and b in the second *i.e.,* $32a + b$. Since $7 + 7 < 16$ we can safely add such bytes together as one nibble can never overflow into the next, *e.g.,*

$$
\begin{aligned}
(2,6) + (3,1) &\to \texttt{00100110} + \texttt{00110001} & (2,6) + (6,6) &\to \texttt{00100110} + \texttt{01100110} \\
&= \texttt{01010111} & &= \texttt{10001100} \\
&\to (5,7) & &\to (8,12)
\end{aligned}
$$

To perform the modular reduction on a byte we first add the byte `00010001` to make 7 overflow to 8 in each nibble. Now take the bitwise `and` of this byte and `10001000`, subtract it, shift it 4 places to the right, then add it on (this performs reduction modulo 8). Finally subtract `00010001`. *e.g.*,

$$\begin{aligned}
\texttt{01010111} + \texttt{00010001} &= \texttt{01101000} \\
\texttt{01101000 and 10001000} &= \texttt{00001000} \\
\texttt{01101000} - \texttt{00001000} &= \texttt{01100000} \\
\texttt{00001000} \xrightarrow{4} &= \texttt{00000001} \\
\texttt{01100000} + \texttt{00000001} &= \texttt{01100001} \\
\texttt{01100001} - \texttt{00010001} &= \texttt{01010000} \\
&\rightarrow (5,0)
\end{aligned}
\qquad
\begin{aligned}
\texttt{10001100} + \texttt{00010001} &= \texttt{10011101} \\
\texttt{10011101 and 10001000} &= \texttt{10001000} \\
\texttt{10011101} - \texttt{10001000} &= \texttt{00010101} \\
\texttt{10001000} \xrightarrow{4} &= \texttt{00010001} \\
\texttt{00010101} + \texttt{00010001} &= \texttt{00100110} \\
\texttt{00100110} - \texttt{00010001} &= \texttt{00010101} \\
&\rightarrow (1,5)
\end{aligned}$$

This procedure is considerably slower than the old lookup table. However, we can apply it to 32 bits at once instead of just a single byte, thus processing 4 times as many field elements at once and realising a two-fold speed increase. The program files are on the CD at `/monster/mop7.1`.

(2.3.5) Experimental Versions And Ideas

In an experimental driver program (`/monster/mop2/idt.c`) we used a pointer linked tree structure to hold a list of words and `suzel`s. Each vertex stores a word WT, a `suzel`, and pointers to the children WBT and WCT provided such words do not contain B^3 or C^4. We used POSIX Threads to calculate the `suzel`s of the children simultaneously. This realised an approximate 25% speed improvement. However, this is an un-successful experiment because a multiple processor machine we can use each processor more profitably by running two instances of the driver program, processing two lots of data: Multi-threading is a false economy for our purposes.

Chapter 3

Conjugacy Class Representatives For The Monster Group

Conjugacy class representatives form part of the data tabulated in [52] as they have many uses. Representatives have been found for 25 of the 26 sporadic simple groups (see [45, 51]). The remaining case is the Monster which is more difficult because of the difficulties of computing in the group. Using the computer constructions described in the previous chapter we find words in the generators for an element in each conjugacy class of $\mathbb{M}$, up to algebraic conjugacy. We are not able to distinguish the classes of elements of order 27 as they have the same character values and power maps. However, we are able to produce an element in class $27A$ and another element in class $27B$, and corresponding words, by a different method.

(3.1) Distinguishing Conjugacy Classes

The ordinary character of degree 196883 is rational and therefore cannot be used to distinguish algebraically conjugate classes. Moreover, the classes $27A$ and $27B$ have the same character value and power maps, so these classes also cannot be distinguished in this way. It is straightforward to check that, up to these ambiguities, the conjugacy class of an element is determined by its order and the character values of it and its powers. Indeed, it is sufficient to use the reduction of the character value modulo 14, *i.e.*, mod 2 and mod 7.

We obtain our character values from trace calculations performed on explicit elements of $\mathbb{M}$. Note that modulo 2 the 196883-dimensional module splits as $1 + 196882$ and so the traces obtained from calculations have the other parity to the character values tabulated in the ATLAS for χ_2. Modulo 7 there is no such splitting and the traces are the 7-modular reductions of the values of χ_2. Wherever possible the modulo 2 values are used since the calculations are much quicker.

For each order of element in $\mathbb{M}$ we tabulate the traces we must find in order to identify each class of elements of that order. These data allow us to answer a second question: Given an element of order n what information must we calculate in order to determine its conjugacy class?

These "conjugacy class identification data" are displayed in Appendix B. The format is easily read by machine (thus is a little cryptic). For each order n of element in $\mathbb{M}$ there is a header row followed by one line for each class of elements of order n. The header row shows what needs to be calculated: "`2`" for a trace modulo 2, "`7`" for a trace modulo 7, and "`aPb`" for the trace modulo `a` of the `b`th power. The subsequent rows give the traces for the class indicated in the first column. A header row containing an asterisk indicates that classes of order n cannot be distinguished. A header row containing the number `1` indicates that there is only one conjugacy class of elements of order n, so no traces need to be calculated.

(3.2) Searching For Conjugacy Class Representatives

In exact analogy with finding conjugacy class representatives in other sporadic simple groups [45], a random search is conducted. The strategy is to generate a long list of words, calculate the orders of the corresponding elements of $\mathbb{M}$, and then use the conjugacy class identification information to work out in which classes these elements lie.

It is sufficient to find a generator for each conjugacy class of maximal cyclic subgroup *i.e.,* a representative for every class that cannot be obtained by powering up an element of greater order. Excluding the classes of algebraically conjugate elements these are $18E$, $24A$, $24C$, $24D$, $24E$, $24F$, $24G$, $24H$, $24I$, $24J$, $30A$, $30F$, $32A$, $32B$, $36A$, $36B$, $36C$, $36D$, $38A$, $39B$, $40A$, $40B$, $41A$, $42D$, $45A$, $48A$, $50A$, $51A$, $52B$, $54A$, $56A$, $57A$, $60A$, $60B$, $60C$, $60D$, $60E$, $60F$, $66A$, $66B$, $68A$, $70A$, $70B$, $78A$, $78C$, $84A$, $84B$, $84C$, $105A$, $110A$. So we must definitely find representatives for these classes. However, powering up produces words with many Ts which are difficult and slow to work with. Therefore it it preferable to find representatives for as many classes as possible, not just those whose elements generate maximal cyclic subgroups. Of course whenever possible it is preferable to find words in $\langle B, C, E\rangle$ since they are much faster to calculate with.

(3.3) Word Generation Strategies

Product replacement [16] has been shown to produce genuinely random words in exactly our situation, but the word length soon becomes very large which would make our calculations too slow. Instead we conducted a search through words of the form W_1T and W_2 where W_1 is a word in B and C, and W_2 is a word in B, C, and E. Eliminating words containing B^3, C^4, or E^3 we obtained 1·4 million words of shape W_1T and calculated element orders for 160,000 of these. For each element order we produced trace calculation jobs according to the data in Appendix B. We store the results of all these calculations in a database which is used to produce further jobs until we have witnessed all classes of elements of that order. For the elements of order 12 for example we ran the following jobs:

1. Trace modulo 7.
2. If the trace modulo 7 is 5 or 6 then calculate the trace modulo 2.
3. If the trace modulo 7 is 5 and the trace modulo 2 is 0 then calculate the trace modulo 7 of the square.
4. If the trace modulo 7 is 6 and the trace modulo 2 is 0 then calculate the trace modulo 7 of the cube.

Representatives for most of the conjugacy classes were found in this way using words of the form W_1T. A few classes of elements of orders 24 and 36 did not turn up and required more sophisticated methods. These classes were $24C$, $24G$, $24F$, $24J$, and $36B$. In fact only 43 elements of the form W_1T have been found in classes $24J$, $24G$, $24F$ and $24E$ (most of them in $24E$) but finding these took over 60 days of computer time.

(3.3.1) Example: Finding An Element In Class 24C

Let $x \in 24C$ then $x^8 \in 3B$ so $C_{\mathbb{M}}(x)$ is the centraliser in $3^{1+12}{\cdot}2{\cdot}\mathrm{Suz}$ of x^3. Since x^3 has order 8 we may think of it as being an element of 2·Suz, which has the following conjugacy classes of elements of order 8:

- Class $8A$ with centraliser in 2·Suz of order $192 = 2^6.3$.
- Classes $+8B$ and $-8B$ both with centraliser in 2·Suz of order $2.64 = 2^7$.
- Class $8C$ with centraliser in 2·Suz of order $32 = 2^5$.

Since $|C_{\mathbb{M}}(x)| = 3456 = 2^7.3$ and 3^{1+12} is a 3-group, the 2-powers in these centraliser orders show that x^3 must lie above class $8B$ of Suz.

Now, C and $B = D$ are standard generators for 6·Suz:2 and standard generators for 6·Suz are A and B where

$$A = (CDD)^{-2}(CD)^{14}(CDD)^2$$

as in [29]. Using A and B in the recipe in [52] for the conjugacy classes of Suz yields elements of 6·Suz lying above the corresponding classes of Suz. From this we obtain elements g_8 lying above $8B$ of order 8, and g_{10} lying above $10B$ of order 60. Therefore $g_6 = g_{10}^{10}$ is a central element of order 6 in 6·Suz. To find all classes lying above $8B$ we calculate the order and traces for $g_6^i g_8$ for $0 \leqslant i \leqslant 5$ and thus find elements of $\mathbb{M}$ in classes $8A$, $8C$, $24C$, and $24I$.

As a word in B and C the length of the element in $24C$ is 161455 which is too long for our results table and takes about 30 minutes to make on machine (using a naïve algorithm). In A and B this is reduced to 205. Looking at elements lying above class $8C$ of Suz yields words in $24G$ and $8C$, but we already have good words for these classes.

(3.4) Elements of Order 27

Class $27A$ has the same character values and power maps as does class $27B$, so they cannot be distinguished mechanically using our method. However, $|C_{\mathbb{M}}(27A)|$ is divisible by 2 whereas $|C_{\mathbb{M}}(27B)|$ is not, and we can exploit this to find words in $\langle B, C, E\rangle$ for elements in each of these classes.

To find such elements we work in the three-fold cover G of $N(3B)$. This is a group of shape $G \cong 3^{1+12}$:6·Suz:2 and it has a representation in 38 dimensions over $\mathbb{F}_3$ that was constructed in [29]. Using this representation we find words in $\langle B, C\rangle$ for an element g of order 9 in 6·Suz:2. There are 3 such conjugacy classes in 6·Suz:2 and they all project onto the same conjugacy class in 2·Suz:2, so we may choose whichever we like.

Now extending downwards to G, we find two conjugacy classes of elements of order 27, and four classes of elements of order 9. One of these classes of elements of order 27 must become $27A$ in the Monster and the other must become $27B$. This is because these two classes are the only two classes of elements of order 27 in the quotient group $N(3B)$. We compute $C_G(g)$ to find which is centralised by an involution, and thus distinguish the classes.

Straight line programs for these elements are included on the CD at `/monster/27`.

Note that an element in class $27A$ can also be obtained by squaring an element in class $54A$.

(3.5) Results

The results consist of a list of conjugacy class names with corresponding words. The table below gives for each conjugacy class of $\mathbb{M}$ the shortest word we found for an element in that class.

The class names are in the second column, and in the third column is the number of words found in the class, if there is a sum then the summands give the number of words of the form W_1T and W_2 respectively. The fourth column gives either a word, or where no element has been found the name of a class which powers up to the particular class, followed in brackets by the number of Ts the power has.

If there exists an element g that powers up to class $3B$ then the class of g, and the classes of all of the powers of g intersect $N(3B)$ non-trivially. Such conjugacy classes are indicated by a bullet in the first column. Of course this test does not identify all of the conjugacy classes of $\mathbb{M}$ that are represented in $\langle B, C, E\rangle$. However, work in Chapter 5 allows us to determine the class fusion from $N(3B)$ to $\mathbb{M}$, and on page 5.6 there is a list of all of the classes of $\mathbb{M}$ that intersect $N(3B)$ non-trivially.

Table 1: Words for conjugacy class representatives

$\langle B, C, E\rangle$	Class	Number	Word
•	$2B$	8	$6C(0)$
•	$2A$	0	$4B(0)$
•	$3B$	0	$6D(0)$
	$3A$	1	B
	$3C$	1	$BCCCBBCBCBCBCBCBBCBBC-$
			$-BBCCCBBCBCBCBCBCBBCBBC-$
			$-BCBBCBBCBBCBBC$
•	$4D$	6982	$BCCCBB$
•	$4A$	8	$BCCBBT$ or $24A(0)$
•	$4B$	84	$BCCCBBCBCBCBCBCBBCBBCB$
•	$4C$	0	$8D(0)$
•	$5A$	14	$BCCCBCCBCBBCBBCBCBCBCBC$

continued...

...continued

$\langle B, C, E \rangle$	Class	Number	Word
•	$5B$	0	$10C(0)$
•	$6B$	0	$12F(4),\ 18A(9),\ 18D(3),\ 18E(3)$
	$6C$	$1052 + 27$	$BCCB$
	$6A$	0	$12C(2),\ 18B(3),\ 18C(6)$
•	$6D$	0	$12G(0)$
•	$6E$	2270	$ECCB$
	$6F$	0	$12D(4)$
•	$7B$	118	$BCCCBCCBC$
	$7A$	0	$14A(6),\ 14B(4),\ 21A(3),\ 21C(6)$
•	$8A$	0	$16A(0),\ 24B(3)$ and see page 23 of text
•	$8D$	0	$24D(0)$
•	$8E$	0	$16B(4),\ 16C(4),\ 24I(0)$
	$8B$	14	$BBCBBCCBCBCBBCBBCBCCCBCBCT$
			or $24A(0)$
•	$8F$	522	$BCCCBCBCBCBCBCBC$
•	$8C$	573	$BCCCBCBCBBCBCBBCBCBC$
•	$9A$	0	$18B(2)$
•	$9B$	55	$BCCCBCCBCBCBCBCBCB$
	$10A$	0	$20B(4),\ 30B(3)$
•	$10E$	0	$20C(6),\ 20F(2),\ 30G$
•	$10C$	170	$BCCCBCBCBCBBCBCBCBCB$
•	$10B$	6383	$BCCCBCBBCBCBCBBCBC$
•	$10D$	0	$20E(2)$
•	$11A$	0	$22A(2),\ 22B(0)$
	$12A$	0	$24A(0),\ 60B(5)$
•	$12I$	368	$BCCCBCBCBCCCBCBBCBBCBCB$
•	$12G$	105	$BCCCBCBBCBCBCBBCBCBBCB$
•	$12F$	0	$24F(2),\ 60E(5)$
•	$12B$	7	$BBCBBCBBCBCBCBBCCCBCBCBT$
	$12E$	0	$24D(0)$
	$12J$	0	$24J(2),\ 60F(5)$

continued...

...continued

$\langle B,C,E\rangle$	Class	Number	Word
	$12D$	0	$24E(2)$
•	$12H$	$0+2$	$24H(0)$
	$12C$	419	$BCCBCCBCBCCCT$
	$13A$	0	$26A(2)$
•	$13B$	54	$BCCCBCCCBCBCBCCBCBCBCBC$
•	$14C$	$116+2$	$BCCCBCCCBCBC$
	$14B$	0	$28B(2)$
	$14A$	0	$28A(6),\ 42A(3)$
	$15A$	0	$30B(2)$
	$15D$	0	$30E(2)$
•	$15B$	0	$30D(2)$
•	$15C$	3	$BCBCBCBCBBCBBCBCT$
	$16A$	$48+1$	$BCCCBBCBCBCBCBCBBCBBC-$
			$-BBCCCBBCBCBCBCBCBBCBBC-$
			$-BBCCCBCBCBBCBCBCBCBCBBCB$
	$16B$	0	$32A(2)$
	$16C$	0	$32B(2)$
	$17A$	55	$BCBCBCBCBCBCBCBT$
•	$18A$	0	$54A(3)$
•	$18E$	26	$BBCBCBCBBCBCCT$
•	$18D$	48	$BCBCBCBCCCBBT$
•	$18B$	57	$BCBBCBBCBCBBT$
•	$18C$	0	$36D(2)$
	$19A$	21	$BBCBBCBBCBBCCCBBCBCBCBCBBT$
	$20A$	0	$40B(2)$
	$20B$	0	$40A(2)$
•	$20C$	0	$60C(3)$
•	$20D$	12	$BCBCBBCBCBBCBCCT$
	$20E$	4	$BBCBBCBCBCBCBBCBCBBT$
•	$20F$	4	$BBCBBCBCBBCBBCT$
	$21C$	0	$42C(2)$

continued...

...continued

$\langle B, C, E\rangle$	Class	Number	Word
	21A	3	$BBCBBCCBCBCBBCBCBCBBCBCT$
•	21D	1	$BBCBBCBBCBCBBCBCBCBCT$
	21B	1	$BBCBBCBBCBBCBBCBBCBCBBT$
•	22B	9	$BCCCBCBCBCBCBCBCBCBCBCBCB$
	22A	21	$BBCBBCBBCBCBT$
	23AB	2361	$BCCBCCT$
•	24H	623	$BCCCBCBCBCBCBBCBCBCBC$
	24A	1	$BCBCBBCBCBCBBCBCT$
			and see page 22 of text
	24B	1	$BCBCBBCBCBCBBCBCBT$
•	24I	31	$BBCBBCBCBCBCBCBBT$
	24D	$222 + 23$	$BCCCBCBCBBCBCBCBCBCBBCB$
•	24C	0	see page 23 of text
	24J	1	$BCBCBCBCBBCBCBCCCT$
•	24G	9923	$EBBCBCBCBCBC$
•	24F	9	$BBCBBCBCBCBBCBBCBCBT$
	24E	33	$BBCBCBCBCCCBT$
	25A	949	$BBCBBCBCBBCBBT$
	26A	8	$BBCBBCBCBCBCCCBCBBCBCCT$
•	26B	78	$BBCBCCCBCBCBBT$
•	27A	0	54A(2)
•	27AB	454	$BBCCBCBCBCBBCT$
•	28D	47	$BBCCCBCBCBCCBCBT$
	28B	27	$BBCCCBT$
	28A	0	84A(3)
	28C	31	$BCBBT$
	29A	1553	$BCCBCBCBBCBCT$
	30B	1	$BBCBBCCBCBCBBCBCBCBBCBCCCT$
	30C	5	$BBCBBCBCBCBCBCBCBBCBCBBBT$
	30E	107	$BCBCBBCBCBBCBBCT$
•	30D	2	$BBCBBCBCBCBCBT$

continued...

...continued

$\langle B, C, E\rangle$	Class	Number	Word
•	$30G$	1	$BBCBCBCBBCBBCT$
•	$30F$	13	$BBCCCBCBCT$
•	$30A$	1	$BCBBCBCBCBBCCT$
	$31AB$	3034	$BBCT$
	$32B$	3	$BCBBCCCT$
	$32A$	1	$BCBCBCBCCT$
	$33B$	5	$BBCBBCCBBCBBCBCBCBBCBCT$
•	$33A$	35	$BBCBCBBCBBCBCBT$
	$34A$	3221	$BCCBCBCBT$
	$35B$	1316	$BCBCBCCBCBT$
	$35A$	0	$70A(2)$
•	$36C$	3	$BCBCBBCBCBBCBBT$
•	$36D$	590	$BCBCBBT$
•	$36A$	1	$BBCCBCBCBCBCBCCCT$
•	$36B$	1	$BBCBBCBBCBBCBBCBCBCBBT$
	$38A$	1628	$BBCBCBCBCBT$
	$39A$	102	$BCBBCBCBCCT$
	$39B$	3	$BBCCBCBBCBBCBCT$
•	$39CD$	44	$BBCBCBBBT$
	$40A$	18	$BBCBBCBBCBCT$
	$40B$	1	$BBCBBCBCBBCBBCBCT$
	$40CD$	688	$BCCCBCBBT$
	$41A$	4949	$BCCT$
	$42C$	75	$BCBBCBCBBCBCBT$
	$42A$	1	$BBCBBCBBCCCBCBBCBCBCBCBT$
•	$42B$	1	$BCBCBCBBCBCBBCBCT$
	$42D$	43	$BCBCBCCBCBCT$
	$44AB$	1657	$BCBCBCBCBCBT$
•	$45A$	2049	$BCBT$
	$46CD$	21	$BBCCCT$

continued...

...continued

$\langle B, C, E\rangle$	Class	Number	Word
	$46AB$	5	$BBCBBCBCT$
	$47AB$	4670	$BCBCBCBBT$
	$48A$	617	$BCBBCBCBBCBBCT$
	$50A$	3342	$BBCBBCCCT$
	$51A$	1102	$BCBBCBCBBCT$
	$52A$	5	$BBCCCBCBCBCBCT$
	$52B$	33	$BBCBCBBT$
•	$54A$	2614	$BCBBCBCBBCBBT$
	$55A$	2035	$BBCBBCBCBBCBCT$
	$56BC$	8	$BBCBBCBCCBCBBCBBT$
	$56A$	6	$BBCCBBCBCBBBT$
	$57A$	1744	$BCCCT$
	$59AB$	3573	$BCBCBCBBCCCT$
	$60A$	4	$BCBBCBCBCCBCBCT$
	$60B$	1	$BCBCBCBCBCBBT$
	$60F$	80	$BBBCT$
•	$60E$	35	$BCBCCBCBCT$
•	$60C$	5	$BBCBCCCBCBBCT$
•	$60D$	2	$BCBCBCBCBCCCT$
	$62AB$	7404	$BCBCBT$
	$66A$	5	$BBCBBCBCBBCBCCBCBBCBCBCT$
•	$66B$	290	$BBCBCBBCBCBT$
	$68A$	2273	$BCBCBCBCT$
	$69AB$	3406	$BBCBCCCT$
	$70B$	3	$BBCBBCBCBBCBCBBCT$
	$70A$	41	$BBCBCBCBCT$
	$71AB$	2588	$BCBBCBCBCT$
•	$78BC$	14	$BBCBCBBCCBBBT$
	$78A$	30	$BBCBCBCBBBT$
	$84C$	2	$BBCBBCBBCBBCBCCCT$
	$84A$	3	$BBCBBCBBCBCBBCBCCT$

continued...

...continued

$\langle B, C, E \rangle$	Class	Number	Word
•	$84B$	8	$BBCBBCBBCBCCBBT$
	$87AB$	3521	$BCBBCBCT$
	$88AB$	2130	$BCBCBCCBCT$
	$92AB$	5731	$BBCBCBCBT$
	$93AB$	6910	$BCBBCBBBT$
	$94AB$	1452	$BCCCBCBCT$
	$95AB$	2313	$BCBCBBCBBBT$
	$104AB$	1367	$BCCCBCBCBCT$
	$105A$	888	$BBCCBCBCBCBCBCBT$
	$110A$	238	$BCBCBBCBCBCBCBBBT$
	$119AB$	3248	$BBCBBCBBBT$

(3.6) Monster Element Database

Due to the large number of words being used it became obvious early on that some sort of database was needed. We store elements and their trace information in text files, one file for each order of elements in $\mathbb{M}$. A suite of Perl scripts manage the database, in particular they can determine what needs to be calculated and produce input data files for the trace finding programs.

The main database programs and data are included on the CD at `/monster/elts` and `/monster/Suzuki`. See Appendix A.2 for an explanation of how to use them.

Chapter 4

Fischer Matrices And The Character Tables Of Groups Of Shape $2^{2} \cdot G{:}2$

"Fischer matrices", "Fischer–Clifford matrices", or, as Fischer calls them, "Clifford matrices" consist of certain coefficients which allow the character table of a group of shape $N.G$ to be assembled from knowledge of certain subgroups of G and the action of G on Irr N.

Originally we were interested in computing the character table of $3^{1+12} \cdot 2 \cdot \mathrm{Suz}{:}2$. Knowing the character table of this group and the class fusion to the Monster would have been of considerable assistance in finding conjugacy class representatives for $\mathbb{M}$: we could have performed structure constant calculations in $N(3B)$. This is now the subject of the next chapter. As an easier example we attempted the character table of $2^{2} \cdot \mathrm{Fi}_{22}{:}2$ which is a maximal subgroup of Fi_{24}. It is not hard to generalise this to the case $2^{2} \cdot G{:}2$, which we do in this chapter.

Most of the required character theory can be found in chapters 6 and 11 of Isaacs's book [21]. Many examples of calculations using Fischer matrices can be found in the theses of Livingstone's students in Birmingham, in particular Jamshid Moori [34] ($2^{10}{:}\mathrm{M}_{22}$ and $2^{10}{:}(\mathrm{M}_{22}{:}2)$) who still works on the subject. The largest group we know of to have had its character table calculated by this method is $2^{1+24} \cdot \mathrm{Co}_1$, the $2B$ involution centraliser in the Monster. Fischer and Livingstone had performed most of the calculations, allowing Thorne to produce the complete character table. However, the only surviving evidence seems to be a brief note in Thorne's thesis [46], and the character table was lost until Norton computed it again (by the same method) in 2001 [40].

Until recently the only reference easily available explaining Fischer matrices was Fischer's rather cryptic paper [11]. Some of this material is explained in rather more detail in some of the theses of Livingstone's students in Birmingham. (We found [44] rather useful.) However, in their recent papers [1, 35] Moori *et al.* have stated and proved again most of the results we require in a rather more accessible way. Our Theorem 12 adds to this effort.

(4.1) Some Clifford Theory

We begin with some notation, terminology, and a summary of some easy results.

Let $\overline{G} = N.G$ be a group, so $N \trianglelefteq \overline{G}$ and the quotient $\overline{G}/N$ is isomorphic to G. In the case of a split extension G is also a subgroup of $\overline{G}$. (This notation is standard in the references given above.)

Let $U \leqslant \overline{G}$, let ϕ be a class function of U and let χ be a class function of $\overline{G}$. Then:

- Let $\dot{\phi}$ be the function on $\overline{G}$ that takes value $\phi(g)$ if $g \in U$ and 0 otherwise.
- Let $\phi{\uparrow}\overline{G}$ be the class function of $\overline{G}$ induced from ϕ (see equation (2) below).
- Let $\chi{\downarrow}U$ be the class function of U restricted from $\overline{G}$.

If $X \subseteq \overline{G}$ and $f \colon X \to \mathbb{C}$ is a class function then for $g \in \overline{G}$ define $f^g(x) = \dot{f}(x^g)$. With this notation the formula for induction of class functions becomes

$$\phi{\uparrow}\overline{G}(x) = \frac{1}{|U|} \sum_{g \in G} \phi^g(x) \tag{2}$$

Define the inertia group of f in $\overline{G}$ by $T_{\overline{G}}(f) = \{t \in \overline{G} \mid f^t = f\}$.

It follows that $\overline{G}$ acts on $\operatorname{Irr} N$ by conjugation. However, since the action of N is trivial, this may be thought of as an action of G.

Theorem 3 (Clifford, Theorem 6.2 in [21]) *Let $\chi \in \operatorname{Irr}\overline{G}$ and let $\theta \in \operatorname{Irr} N$ be a constituent of $\chi{\downarrow}N$. Suppose that $\theta = \theta_1, \theta_2, \ldots, \theta_t$ are the distinct $\overline{G}$-conjugates of θ. Then*

$$\chi{\downarrow}N = \langle \chi{\downarrow}N, \theta \rangle \sum_{i=1}^{t} \theta_i$$

Theorem 4 (Clifford, Theorem 6.11 in [21]) *Let $\theta \in \operatorname{Irr} N$ and let $\overline{H}$ be the inertia group in $\overline{G}$ of θ. Define*

$$\mathcal{U} = \{\psi \in \operatorname{Irr} \overline{H} \mid \langle \psi{\downarrow}N, \theta \rangle \neq 0\}$$
$$\mathcal{V} = \{\chi \in \operatorname{Irr} \overline{G} \mid \langle \chi{\downarrow}N, \theta \rangle \neq 0\}$$

Then induction from $\overline{H}$ to $\overline{G}$ defines a bijection from $\mathcal{U}$ to $\mathcal{V}$.

Theorem 5 (Corollary 6.17 in [21]) *Suppose that $\psi \in \operatorname{Irr} \overline{H}$ is an extension of $\theta \in \operatorname{Irr} N$,* i.e., *$\psi{\downarrow}N = \theta$. Let*

$$\mathcal{W} = \{\eta \in \operatorname{Irr} \overline{H} \mid N \subseteq \ker \eta\}$$

and let $\mathcal{U}$ be as in Theorem 4. Then multiplication by ψ defines a bijection from $\mathcal{W}$ to $\mathcal{U}$.

Corollary 6 (In section 3 of [35]) *Let $\theta_1, \theta_2, \ldots, \theta_t$ be representatives of the orbits of $\overline{G}$ on $\operatorname{Irr} N$, and let $\overline{H}_i$ be the inertia group of θ_i in $\overline{G}$. Suppose that for each i there exists $\psi_i \in \operatorname{Irr} \overline{H}_i$ that is an extension of θ_i. Then*

$$\operatorname{Irr} \overline{G} = \bigcup_{i=1}^{t} \{(\psi_i \eta){\uparrow}\overline{G} \mid \eta \in \operatorname{Irr} \overline{H}_i \text{ and } N \subseteq \ker \eta\}$$

Of course it is not necessarily the case that there exists an extension ψ of θ to the inertia group. However, there is always a *projective* extension:

Definition 7 *Let G be a group and let $\mathfrak{X}\colon G \to \mathbb{C}$ be such that for every $g, h \in G$ there exists a scalar $\alpha(g,h)$ such that $\mathfrak{X}(gh) = \alpha(g,h)\mathfrak{X}(g)\mathfrak{X}(h)$. Then $\mathfrak{X}$ is a projective $\mathbb{C}$-representation of G and the function $\alpha\colon G \times G \to \mathbb{C}$ is the associated factor set of $\mathfrak{X}$.*

Theorem 8 (Theorem 11.2 in [21]) *Let N be a normal subgroup of a group G and let $\mathfrak{N}$ be an irreducible $\mathbb{C}$-representation of N whose character is invariant in G. Then there exists a projective $\mathbb{C}$-representation $\mathfrak{X}$ of G such that for all $n \in N$ and $g \in G$ we have*

(i) $\mathfrak{X}(n) = \mathfrak{N}(n)$.

(ii) $\mathfrak{X}(ng) = \mathfrak{X}(n)\mathfrak{X}(g)$.

(iii) $\mathfrak{X}(gn) = \mathfrak{X}(g)\mathfrak{X}(n)$.

We extend such a representation $\mathfrak{N}$ of N to a projective representation $\mathfrak{X}$ of $\overline{H}_i$ as follows: Take a transversal T for N in $\overline{H}_i$ and for $t \in T$ choose a non-singular matrix P_t with $P_t \mathfrak{N} P_t^{-1} = \mathfrak{N}^t$ and define $\mathfrak{X}(nt) = \mathfrak{N}(n)P_t$. The matrix P_t is only defined up to multiplication by scalars so we obtain $P_tP_r = \alpha(t,r)P_{tr}$ for some scalar $\alpha(t,r)$. (Here α is a 2-coboundary, and even when the extension $N.H_i$ is split, α is not necessarily (equivalent by a 2-cocycle to) 1.)

When the extension $\mathfrak{X}$ is projective Theorem 5 and Corollary 6 generalise (as is shown, for example, in [25]) with ψ_i the projective extension of θ_i and η an irreducible projective character of $\overline{H}_i$.

(4.2) Conjugacy Classes

For $g \in G$ let $\overline{g} \in \overline{G}$ map to g under the natural homomorphism, *i.e.*, $g = N\overline{g}$. Each conjugacy class of $\overline{G}$ corresponds (under the natural homomorphism) to a unique conjugacy class of G. (The coset representatives are conjugate in $\overline{G}$ therefore the cosets must be conjugate in G.)

Lemma 9 *For any conjugacy class representative $g = N\overline{g}$ write*

$$g = N\overline{g} = (N\overline{g} \cap [x_1]_{\overline{G}}) \,\dot{\cup}\, (N\overline{g} \cap [x_2]_{\overline{G}}) \,\dot{\cup} \ldots \dot{\cup}\, (N\overline{g} \cap [x_{c(g)}]_{\overline{G}}) \tag{10}$$

then $\{x_1, x_2, \ldots, x_{c(g)}\}$ is a complete set of representatives for the conjugacy classes of $\overline{G}$ that correspond (under the natural homomorphism) to $[g]_G$.

Proof. Let $\overline{h} \in \overline{G}$ have image $h \in G$ that is G-conjugate to g. We must show that $\overline{h}$ is $\overline{G}$-conjugate to one of the x_j. Suppose that $k \in G$ conjugates h to g then some pre-image $\overline{k}$ of k has $\overline{h}^{\overline{k}} \in N\overline{g}$ and so $\overline{h}^{\overline{k}}$ must be $\overline{G}$-conjugate to one of the x_j. □

Corollary 11 *The conjugacy class $[x_j]_{\overline{G}}$ is partitioned into $|[g]_G|$ subsets of equal size, each set of the partition being the intersection of $[x_j]_{\overline{G}}$ with an element of $[g]_G$.*

Therefore information about every conjugacy class of $\overline{G}$ can be obtained by examining one coset $g \in G$ for each conjugacy class of G. Choose $x_1 = \overline{g}$.

The set $[g] \cap H_i$ can be partitioned into H_i-conjugacy classes, say

$$[g] \cap H_i = [y_1]_{H_i} \,\dot{\cup}\, [y_2]_{H_i} \,\dot{\cup} \ldots \dot{\cup}\, [y_r]_{H_i}$$

which fuse in G. Now, each y_k is really a coset of N by some element in $\overline{H}_i$. By the same argument as in Lemma 9, each $[y_k]$ corresponds to a number of conjugacy classes of $\overline{H}_i$. Let $\{y_{l_k}\}$ be a set of

$$
\begin{tikzcd}
\overline{G} \arrow[d, dash] \arrow[dr, dash] & \\
\bigcup_{k,l} [y_{l_k}] = \overline{H} \arrow[d, dash] \arrow[dr, dash] & G \ni N\overline{g} = g \in [g] \arrow[d, dash] \\
N & H_i = [y_1] \dot{\cup} [y_2] \dot{\cup} \ldots \dot{\cup} [y_k]
\end{tikzcd}
$$

Figure 1: Some groups

representatives for these conjugacy classes. The situation is illustrated in Figure 1. The y_{l_k} will be used to index the rows of the Fischer matrices.

(4.3) The Fischer Matrices

The previous section collects the information needed about the conjugacy classes of $\overline{G}$. Observe that since $N \subseteq \ker \eta$, there is an irreducible character $\hat{\eta} \in \operatorname{Irr} H_i$ which lifts to η, *i.e.*, $\eta(y_{l_k}) = \hat{\eta}(y_k)$ for all l. Enough information has now been gathered to re-write the character induction formula for Corollary 6.

Theorem 12 (In [11, 35]) *With notation as above,* $(\psi_i\eta)\uparrow\overline{G}(x_j) = \sum_{k=1}^{r} \hat{\eta}(y_k) \sum_{\{l \mid y_{l_k} \sim_{\overline{H}_i} x_j\}} \dfrac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|} \psi_i(y_{l_k})$.

Proof. The induction formula (2) for chosen i and j gives

$$(\psi_i\eta)\uparrow\overline{G}(x_j) = \frac{1}{|\overline{H}_i|} \sum_{\{\overline{h} \in \overline{G} \mid x_j^{\overline{h}} \in \overline{H}_i\}} \psi_i\eta(x_j^{\overline{h}})$$

Now, the classes $[y_{l_k}]_{\overline{H}_i}$ partition $\overline{H}_i$, so the statement $x_j^{\overline{h}} \in \overline{H}_i$ is equivalent to $x_j^{\overline{h}} \sim_{\overline{H}_i} y_{l_k}$ for those y_{l_k} that are $\overline{G}$-conjugate to x_j, *i.e.*,

$$(\psi_i\eta)\uparrow\overline{G}(x_j) = \frac{1}{|\overline{H}_i|} \sum_{k=1}^{r} \sum_{\{l \mid y_{l_k} \sim_{\overline{G}} x_j\}} \sum_{\{\overline{h} \in \overline{G} \mid x_j^{\overline{h}} \sim_{\overline{H}_i} y_{l_k}\}} \psi_i\eta(x_j^{\overline{h}})$$

But $\psi_i\eta$ is a class function of $\overline{H}_i$ and so its value is constant over the range of the rightmost sum above.

Hence

$$\begin{aligned}(\psi_i\eta)\!\uparrow\!\overline{G}(x_j) &= \frac{1}{|\overline{H}_i|}\sum_{k=1}^{r}\sum_{\{l|y_{l_k}\sim_{\overline{G}}x_j\}} |\{\overline{h}\in\overline{G} \mid x_j^{\overline{h}} \sim_{\overline{H}_i} y_{l_k}\}|\psi_i\eta(y_{l_k}) \\ &= \frac{1}{|\overline{H}_i|}\sum_{k=1}^{r}\sum_{\{l|y_{l_k}\sim_{\overline{G}}x_j\}} |[y_{l_k}]_{\overline{H}_i}||\{\overline{h}\in\overline{G} \mid x_j^{\overline{h}} = y_{l_k}\}|\psi_i\eta(y_{l_k}) \\ &= \frac{1}{|\overline{H}_i|}\sum_{k=1}^{r}\sum_{\{l|y_{l_k}\sim_{\overline{G}}x_j\}} |[y_{l_k}]_{\overline{H}_i}||\{\overline{h}\in\overline{G} \mid x_j^{\overline{h}} = x_j\}|\psi_i\eta(y_{l_k}) \\ &= \sum_{k=1}^{r}\sum_{\{l|y_{l_k}\sim_{\overline{G}}x_j\}} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i\eta(y_{l_k}) \\ &= \sum_{k=1}^{r}\hat{\eta}(y_k)\sum_{\{l|y_{l_k}\sim_{\overline{G}}x_j\}} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k}) \qquad \square\end{aligned}$$

Note that by writing

$$a_{kj}^{(i)} = \sum_{\{l|y_{l_k}\sim_{\overline{G}}x_j\}} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|}\psi_i(y_{l_k}) \tag{13}$$

Theorem 12 becomes

$$\psi_i\eta\!\uparrow\!\overline{G}(x_j) = \sum_{k=1}^{r} a_{kj}^{(i)}\hat{\eta}(y_k)$$

and write $M_i(g) = (a_{kj}^{(i)})$. This will become part of the Fischer matrix $M(g)$ for g. (Recall that $g \in G$ is the coset of N in which the conjugacy classes of $\overline{G}$ with representatives $x_1, x_2, \ldots, x_{c(g)}$ are visible.) So,

$$\begin{aligned}&\begin{pmatrix}\hat{\eta}(y_1) & \hat{\eta}(y_2) & \ldots & \hat{\eta}(y_r)\end{pmatrix}\begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \ldots & a_{1,c(g)}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} & \ldots & a_{2,c(g)}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1}^{(i)} & a_{r,2}^{(i)} & \ldots & a_{r,c(g)}^{(i)}\end{pmatrix} \\ &\qquad = \begin{pmatrix}(\psi_i\eta)\!\uparrow\!\overline{G}(x_1) & (\psi_i\eta)\!\uparrow\!\overline{G}(x_2) & \ldots & (\psi_i\eta)\!\uparrow\!\overline{G}(x_{c(g)})\end{pmatrix}\end{aligned} \tag{14}$$

The Fischer matrix of $\overline{G}$ corresponding to the coset $N\overline{g}$ is defined as

$$M(g) = \begin{pmatrix} M_1(g) \\ M_2(g) \\ \vdots \\ M_t(g)\end{pmatrix}$$

where if $H_i \cap [g] = \emptyset$ then $M_i(g)$ is not defined and is omitted from $M(g)$.

Let $R(g) = \{(i, y_k) \mid 1 \leqslant i \leqslant t, H_i \cap [g] \neq \emptyset, 1 \leqslant k \leqslant r\}$. Then clearly the number of rows in $M(g)$ is $|R(g)|$. Hence $M(g) = (a_j^{(i,y_k)})$ where

$$a_j^{(i,y_k)} = \sum_{\{l \mid y_{l_k} \sim_{\overline{G}} x_j\}} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|} \psi_i(y_{l_k})$$

cf. equation 1.10.5 in [11].

Now suppose that m characters of $\overline{H}_i$ have N in their kernel (equivalently, H_i has m irreducible characters), then equation (14) becomes

$$\begin{pmatrix} \hat{\eta}_1(y_1) & \hat{\eta}_1(y_2) & \dots & \hat{\eta}_1(y_r) \\ \hat{\eta}_2(y_1) & \hat{\eta}_2(y_2) & \dots & \hat{\eta}_2(y_r) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\eta}_m(y_1) & \hat{\eta}_m(y_2) & \dots & \hat{\eta}_m(y_r) \end{pmatrix} \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \dots & a_{1,c(g)}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} & \dots & a_{2,c(g)}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1}^{(i)} & a_{r,2}^{(i)} & \dots & a_{r,c(g)}^{(i)} \end{pmatrix}$$
$$= \begin{pmatrix} (\psi_i\eta_1)\uparrow\overline{G}(x_1) & (\psi_i\eta_1)\uparrow\overline{G}(x_2) & \dots & (\psi_i\eta_1)\uparrow\overline{G}(x_{c(g)}) \\ (\psi_i\eta_2)\uparrow\overline{G}(x_1) & (\psi_i\eta_2)\uparrow\overline{G}(x_2) & \dots & (\psi_i\eta_2)\uparrow\overline{G}(x_{c(g)}) \\ \vdots & \vdots & \ddots & \vdots \\ (\psi_i\eta_m)\uparrow\overline{G}(x_1) & (\psi_i\eta_m)\uparrow\overline{G}(x_2) & \dots & (\psi_i\eta_m)\uparrow\overline{G}(x_{c(g)}) \end{pmatrix}$$

or, more concisely, write K_i for the fragment of the character table of H_i consisting of the columns of the conjugacy classes corresponding to $[g]$. If there is no complementary H_i then use the projective character table instead. Then the fragment of the character table of $\overline{G}$ consisting of the columns of the conjugacy classes corresponding to $[g]$ is

$$\begin{pmatrix} K_1 M_1(g) \\ K_2 M_2(g) \\ \vdots \\ K_t M_t(g) \end{pmatrix}$$

By analysing the cosets of N more carefully, more information can be added to the Fischer-Clifford matrices. This is the subject of following sections.

(4.4) Coset Analysis And Properties Of Fischer Matrices

In Section 4.2 we observed that representatives for the conjugacy classes of $\overline{G}$ can be found amongst the elements of the cosets $N\overline{g}$ which form a set of class representatives for G.

Suppose that N is elementary Abelian and let $C_{\overline{g}}$ be the setwise stabiliser in $\overline{G}$ of the coset $N\overline{g}$. It is clear (Lemma 2.4 in [35]) that

$$\frac{C_{\overline{g}}}{N} = C_{\frac{\overline{G}}{N}}(N\overline{g})$$

and this may be identified with $C_G(g)$ via the natural isomorphism. Now, clearly $C_{\overline{g}} = N.C_G(g)$ and so the action of $C_{\overline{g}}$ on $N\overline{g}$ can be examined as follows.

Firstly, "STEP 1" of [35] shows that $C_N(n\overline{g}) = C_N(\overline{g})$ for any $n \in N$, since N is Abelian, so $C_N(\overline{g})$ fixes $N\overline{g}$ pointwise. Hence under the action of conjugation by elements of N, all point stabilisers are $C_N(\overline{g})$ and hence the orbits, $Q_1, Q_2, \ldots, Q_k$ say, all have length $[N : C_N(\overline{g})]$ and $k = |C_N(\overline{g})|$.

Secondly, elements $\{\overline{h} \mid h \in C_G(g)\}$ act to fuse some of the N-orbits, Q_i. Suppose that f_j of the Q_i fuse into a new orbit Δ_j then, as in "STEP 2" of [35],

$$|\Delta_j| = f_j \frac{|N|}{k}$$

and these must be the "$N\overline{g} \cap [x_j]_{\overline{G}}$" from equation (10). Now by Corollary 11 for $x_j \in \Delta_j$

$$|[x_j]_{\overline{G}}| = |\Delta_j||[g]_G| = f_j \frac{|N|}{k} \frac{|G|}{|C_G(g)|} = f_j \frac{|\overline{G}|}{k|C_G(g)|}$$

and hence

$$|C_{\overline{G}}(x_j)| = \frac{k|C_G(g)|}{f_j}$$

This information is used to label the Fischer matrices. Let $m_j = [C_{\overline{g}} : C_{\overline{G}}(x_j)]$. The columns of $M(g)$ are indexed by $x_1, x_2, \ldots, x_{c(g)}$, so label the top of $M(g)$ with $|C_{\overline{G}}(x_j)|$ and label the bottom of each column with m_j. The rows of $M(g)$ are already partitioned into blocks corresponding to various inertia groups, so label the left of $M(g)$ with $|C_{H_i}(y_k)|$.

The extra labels allow us to state the following theorem from [11] and [1].

Theorem 15 *The Fischer matrix $M(g)$ has the following properties:*

(a) $a_j^{(1,g)} = 1$ *for all* $1 \leqslant j \leqslant c(g)$.

(b) $|R(g)| = c(g)$.

(c) $\sum_{j=1}^{c(g)} m_j a_j^{(i,y_k)} \overline{a_j^{(i',y_k')}} = \delta_{(i,y_k)(i',y_k')} \dfrac{|C_G(g)|}{|C_{H_i}(y_k)|} |N|$ *(weighted row orthogonality).*

(d) $\sum_{(i,y_k)\in R(g)} a_j^{(i,y_k)} \overline{a_{j'}^{(i,y_k)}} |C_{H_i}(y_k)| = \delta_{jj'} |C_{\overline{G}}(x_j)$ *(weighted column orthogonality).*

(e) $M(g)$ *is a square non-singular matrix.*

Furthermore, if the extension $N.G$ *is split then*

(f) $a_1^{(i,y_k)} = \dfrac{|C_G(g)|}{|C_{H_i(y_k)}|}$ *for all* $(i, y_k) \in R(g)$.

(4.5) Extending G to G:2 (or *vice versa*)

The following theorem is apparent by comparing [1] and [2], but does not appear to be recorded anywhere.

Theorem 16 *Let* G *be a group with an outer automorphism of order 2 and that acts on a vector space* V *so that a split extension* $V{:}(G{:}2)$ *may be formed.*

The Fischer matrices for the extension $V{:}(G{:}2)$ *are related to those of* $V{:}G$ *in the following way: Let* $g \in G \leqslant G{:}2$ *and let* σ *be an involution that extends* G *to* $G{:}2$ *then*

(i) If σ *fuses classes* $[g]$ *and* $[g']$ *of* G *to a single class* $[g]$ *of* $G{:}2$ *then* $M_G(g) = M_{G:2}(g)$ *and the labels of the matrices are also equal.*

(ii) If $[g]_G = [g]_{G:2}$ *and* σ *does not fuse any classes in any of the inertia groups, then* $M_G(g) = M_{G:2}(g)$ *and for* $M_{G:2}(g)$ *all of the bottom column labels are the same. If* g *is centralised by an outer element then the row and top column labels are doubled, otherwise they are as for* $M_G(g)$.

(iii) If $[g]_G = [g]_{G:2}$ *and* σ *fuses some conjugacy classes of the inertia groups then:*

- *On the rows corresponding to classes of the inertia groups that do not fuse under the action of* σ *the entries are the same and the row labels are doubled.*
- *The rows corresponding to the fusing classes are deleted and replaced by a new row which is the coordinate-wise sum of the deleted rows. The new row label is the same as those of the deleted rows.*
- *One of the columns corresponding to the fused classes is deleted and the bottom column label of the remaining column is doubled.*

Proof. Recall that for $V{:}G$ the the top column labels are

$$|C_{\overline{G}}(x_j)| = \frac{k|C_G(g)|}{f_j},$$

the bottom column labels are

$$m_j = [C_{\overline{g}} : C_{\overline{G}}(x_j)] = \frac{|V|f_j}{k},$$

and the row labels are $|C_{H_i}(y_k)|$ where $[y_k]_{H_i}$ fuses to $[g]_G$. The size of the fixed space of g is k, which is clearly the same in G and $G{:}2$.

Suppose that $[g]$ and $[g']$ are conjugacy classes of G that fuse under the action of the outer automorphism, σ. Then σ doesn't centralise g and so $C_{G:2}(g) = C_G(g)$ and there is no action of σ on Vg. Hence the f_j are the same so that the top and bottom column labels do not change. Clearly $C_{H_i:2}(y_k) = C_{H_i}(y_k)$ and so the row labels are not changed. Since

$$\frac{|C_G(g)|}{|C_{H_i}(y_k)|} = \frac{|C_{G:2}(g)|}{|C_{H_i:2}(y_k)|}$$

the conditions of Theorem 15 are unaffected and the matrices must also have the same entries.

If $[g]$ does not fuse with another class, and there is no fusion in any of the inertia groups that extend to $H_i{:}2$ then $M_G(g)$ and $M_{G:2}(g)$ have the same number of rows and thus columns, and thus the f_js must be the same for $M_G(g)$ and $M_{G:2}(g)$. If an outer element centralises g then clearly the top column labels must double, as must the row labels in this case.

Finally, suppose that some of the inertia groups extend and that σ fuses classes in some of these. Then σ fuses two of the classes of $V{:}G$ that lie above $[g]$. Therefore when σ acts on Vg it must fuse two of the previous orbits (which must be the same size). The effect of this is to remove two equal f_js and replace them with one of twice the size. The new labels are therefore justified and it is easy to show that the described contortion of the Fischer-Clifford matrix yields a matrix that obeys the conditions of Theorem 15. In particular, suppose that classes $[y_k]$ and $[k_{k'}]$ of the inertia group H_i are fused by σ into a new class

$[y_m]$. Let $\overline{G}{:}2 = V{:}G{:}2$ then

$$
\begin{aligned}
a_j^{(i,y_m)} &= \sum_{\{l \mid y_{l_m} \overset{\overline{G}:2}{\sim} x_j\}} \frac{|C_{\overline{G}:2}(x_j)|}{|C_{\overline{H}_i:2}(y_{l_m})|} \psi_i(y_{l_k}) \\
&= \sum_{\{l \mid y_{l_k} \overset{\overline{G}}{\sim} x_j\}} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_k})|} \psi_i(y_{l_m}) + \sum_{\{l \mid y_{l_{k'}} \overset{\overline{G}}{\sim} x_j\}} \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H}_i}(y_{l_{k'}})|} \psi_i(y_{l_{k'}}) \\
&= a_j^{(i,k_k)} + a_j^{(i,y_{k'})}
\end{aligned}
$$

$\square$

(4.6) The Character Table Of $2^{2\cdot}$G:2

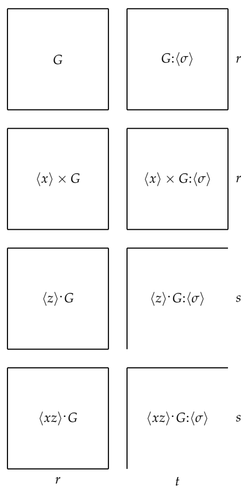
Let G be a finite group with an automorphism of order 2 and a double cover. We may form a group of shape $2^{2\cdot}G{:}2$ which we write as

$$(\langle x \rangle \times \langle z \rangle {}^{\cdot} G){:}\langle \sigma \rangle$$

so that σ acts to swap x with xz and $\langle x, z, \sigma \rangle \cong D_8$. This group has three interesting subgroups of index 2:

$$G^+ = \langle z \rangle {}^{\cdot} G{:}\langle \sigma \rangle \qquad G^- = \langle z \rangle {}^{\cdot} G {}^{\cdot} \langle x\sigma \rangle \qquad G^0 = \langle x \rangle \times \langle z \rangle {}^{\cdot} G$$

Here G^+ and G^- are representatives of the two isomorphism classes of groups of shape $2{}^{\cdot}G.2$ (they are isoclinic). To produce the character table of $2^{2\cdot}G{:}2$ all we must do is determine the class fusion from G^0 and the character values on the outer conjugacy classes. Of course, it is trivial to write down the character table of G^0 given the character table of $2{}^{\cdot}G$. The ATLAS map of the character table of $2^{2\cdot}G{:}2$ is:



Here G has r conjugacy classes, $2{\cdot}G$ has s characters that are faithful on z, and $G{:}2$ has t conjugacy classes of outer elements.

For any character χ in the $\langle z\rangle{\cdot}G$ square we have $\chi^\sigma(x) = \chi(xz) = -\chi(1)$ and $\chi^\sigma(xz) = \chi(x) = \chi(1)$. Such characters only occur in the $\langle xz\rangle{\cdot}G$ square. Thus the $\langle z\rangle{\cdot}G{:}\langle\sigma\rangle$ and $\langle xz\rangle{\cdot}G{:}\langle\sigma\rangle$ squares contain only zeros.

Each outer class $[g]$ of $G{:}2$ lifts to two classes in $2^2{\cdot}G{:}2$, these are $[g, zg]$ and $[xg, xzg]$. To see this observe that $g^x = xgx = xxzg = zg$, which also shows that if the order of g is indivisible by 4 then the order of xg is twice the order of g. A character $\chi \in \operatorname{Irr} G{:}2$ takes value $\chi(xg) = -\chi(g)$, which, it would seem, completes the table.

We wish to improve the above analysis to sufficient precision to allow the character table to actually be produced. There are about 12 different ways in which conjugacy classes of $G{:}2$ can lift to $2^2{\cdot}G{:}2$, and these are explained in the case analysis beginning on page 43 in the next section. (This number is approximate because there are some subtleties involving the element orders which one may use to produce further cases.) We choose to work with the example $G = \mathrm{Fi}_{22}$ because the character table of $2^2{\cdot}\mathrm{Fi}_{22}{:}2$ is not stored in GAP, most of the conjugacy class types occur (the three that don't can be seen in A_5 or $\mathrm{L}_2(17)$), and we use the table briefly in Section 6.7.2.

We compute the character table using Fischer matrices as in this case we can eliminate all uncertainty about the entries of the matrices. Thus assembling the character table is easy. We shall need the character table of $G{:}2$ and the projective character table of G. This is no more information that we needed above, for the projective character table of G is easily deduced from the character table of $2{\cdot}G$.

(4.7) The Character Table Of $2^2{\cdot}\mathrm{Fi}_{22}{:}2$

The non-split extension of $N = 2^2$ by $G = \mathrm{Fi}_{22}{:}2$ provides an easy example of a Fischer matrices calculation. We use information from the character tables of the groups $2{\cdot}\mathrm{Fi}_{22}{:}2$ and $2 \times 2{\cdot}\mathrm{Fi}_{22}{:}2$ to check our calculations.

As in the previous section we write our group $\overline{G} = 2^2{\cdot}\mathrm{Fi}_{22}{:}2 \cong (2 \times 2{\cdot}\mathrm{Fi}_{22}){:}2$ as

$$(\langle x\rangle \times \langle z\rangle{\cdot}\mathrm{Fi}_{22}){:}\langle\sigma\rangle$$

so that σ acts to swap x with xz and $\langle x, z, \sigma \rangle \cong D_8$. $\overline{G}$ has three orbits on elements of the normal 2^2 group and the stabilisers are:

(i) $\overline{G}$ fixing the identity. The trivial character of 2^2 extends to $\overline{G}$.

(ii) $2 \times 2{}^{\cdot}\mathrm{Fi}_{22}$ fixing x and xz. Neither of the other two characters of the 2^2 that represent z faithfully can extend to the inertia group for the following reason: Choose $g \in 2^2{}^{\cdot}\mathrm{Fi}_{22}$ that is conjugate to zg (which is possible since the extension is non-split) and let χ be such an extension. Then $\chi(g) = \chi(zg)$ which is non-zero as χ is linear. But z is represented as -1 which forces $\chi(g) = -\chi(zg)$.

(iii) $\overline{G}$ fixing z. The character that takes values -1 on x and xz extends to the inertia group.

The required inertia factors are therefore $\mathrm{Fi}_{22}{:}2$, Fi_{22}, and $\mathrm{Fi}_{22}{:}2$ respectively, and we must use the projective character table of Fi_{22}.

The program to construct the character table is on the CD at `/Fischer/cl4.gap`. This file is not easy to read, and should be viewed as a draft of `/Fischer/ct22g2.gap` which defines a function
`CharacterTableTwoSquaredGsplitTwo`
for constructing the character table of such a group.

The calculations are described in the following sections.

(4.7.1) Conjugacy Classes and Fischer Matrices

We can easily find the number of conjugacy classes of $2^2{}^{\cdot}\mathrm{Fi}_{22}{:}2$ lying above a class $[g]$ of $\mathrm{Fi}_{22}{:}2$ by counting class fusions from our inertia groups and using the fact that Fischer matrices are square. With a little more work we can also compute the class fusion from $2 \times 2{}^{\cdot}\mathrm{Fi}_{22}$ which we use later to check our calculations.

We consider the diagram in Figure 2(a), and for $g \in 2{}^{\cdot}\mathrm{Fi}_{22}{:}2$ write $\hat{g}$ for the image of g under the natural homomorphism to $\mathrm{Fi}_{22}{:}2$. The following cases arise:

1. Let $[\hat{g}]$ be a conjugacy class of Fi_{22} that does not fuse with another class under the action of σ.
   (a) If g is not conjugate to zg in $2{}^{\cdot}\mathrm{Fi}_{22}$ then:
      i. If g is not conjugate to zg in $2{}^{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 1A$ as in Figure 2(b)) then we obtain three conjugacy classes with representatives g, xg, and zg respectively. (The second column corresponds to a class of twice the size of the other two, so xg and xzg must fuse to this

$$\begin{tikzcd}
\mathrm{Fi}_{22} \arrow[r, "\text{class fusion}"] \arrow[d, "\text{lift}"'] & \mathrm{Fi}_{22}{:}2 \arrow[d, "\text{lift}"] \\
2{\cdot}\mathrm{Fi}_{22} \arrow[r, "\text{class fusion}"] & 2{\cdot}\mathrm{Fi}_{22}{:}2
\end{tikzcd}$$

(a) The groups.

$$\begin{tikzcd}
1A \arrow[r] \arrow[d] & 1A \arrow[d] \\
\begin{matrix} +1A \\ -1A \end{matrix} \arrow[r] & \begin{matrix} +1A \\ -1A \end{matrix}
\end{tikzcd}$$

(b) As in case 1(a)i.

$$\begin{tikzcd}
2A \arrow[r] \arrow[d] & 2A \arrow[d] \\
\begin{matrix} +2A \\ -2A \end{matrix} \arrow[r] & \pm 2A
\end{tikzcd}$$

(c) As in case 1(a)ii.

$$\begin{tikzcd}
16A \quad 16B \arrow[r] \arrow[d] & 16AB \arrow[d] \\
\pm 16A \quad \pm 16B \arrow[r] & \pm 16AB
\end{tikzcd}$$

(d) As in case 2(a).

$$\begin{tikzcd}
11A \quad 11B \arrow[r] \arrow[d] & 11AB \arrow[d] \\
\begin{matrix} +11A & +11B \\ -11A & -11B \end{matrix} \arrow[r] & 11AB \quad -11AB
\end{tikzcd}$$

(e) As in case 2(b).

$$\begin{tikzcd}
18A \quad 18B \arrow[r] \arrow[d] & 18AB \arrow[d] \\
\begin{matrix} +18A & +18B \\ -18A & -18B \end{matrix} \arrow[r] & \begin{matrix} +18A \sim -18B \\ -18A \sim +18B \end{matrix}
\end{tikzcd}$$

(f) As in case 2(c).

Figure 2: The relationship between conjugacy classes of Fi_{22} and $2{\cdot}\mathrm{Fi}_{22}{:}2$. Horizontal arrows are for class fusion under the automorphism, and vertical arrows show lifting to the double cover. Diagram (a) shows the groups involved. The other diagrams give examples of some of the possibilities. Class names are for Fi_{22}.

class.) Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} \quad \text{with} \quad k = 4 \text{ and } f_1 = 1, f_2 = 2, f_3 = 1$$

the class $[\hat{g}]$ lifts to the first of these new classes and these elements have order $|g|$. If g has odd order then the other two classes contain elements of order $2|g|$, otherwise they contain elements of order $|g|$. (It is possible that $|g| = 2|\hat{g}|$. This case does not occur for Fi_{22}, the smallest ATLAS group where it does occur is $\mathrm{L}_2(17)$.)

ii. If g is conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 2A$, as in Figure 2(c)) then we obtain three conjugacy class representatives xg, g, and xzg. Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & -1 \end{pmatrix} \quad \text{with} \quad k = 4 \text{ and } f_1 = 1, f_2 = 2, f_3 = 1$$

the class $[\hat{g}]$ must lift to the second of these new classes which is twice the size of the other two. The orders of the elements in both classes is the same as the order of g.

(b) i. Suppose that g is conjugate to zg in $2{}^{\cdot}\mathrm{Fi}_{22}$ and g has the same order as $\hat{g}$ (*e.g.*, $\hat{g} \in 2C$). Then g and zg are also conjugate in $2{}^{\cdot}\mathrm{Fi}_{22}{:}2$ and xg is conjugate to xzg in $\overline{G}$. We thus obtain 2 conjugacy classes and our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad k = 2 \text{ and } f_1 = 1, f_2 = 1$$

the class $[\hat{g}]$ lifts to the first of these new classes. Both new classes contain elements with the same order as $\hat{g}$.

ii. Suppose that g is conjugate to zg in $2{}^{\cdot}\mathrm{Fi}_{22}$ and g has twice the order of $\hat{g}$. This case does not occur for Fi_{22}. An example is class $2A$ of A_5. The only difference from case 1(b)i is the element orders which are doubled.

2. Let $[\hat{g}]$ be a conjugacy class of Fi_{22} that is fused with a class $[\hat{h}]$ in $\mathrm{Fi}_{22}{:}2$.

(a) If g is conjugate to zg in $2{}^{\cdot}\mathrm{Fi}_{22}$ (*e.g.*, $g \in 16A$ as in Figure 2(d)) then we obtain two conjugacy classes with representatives $g \sim zg \sim h \sim zh$ and $zg \sim xzg \sim xh \sim xzh$. The Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{with} \quad k = 2 \text{ and } f_1 = 1, f_2 = 1$$

and the class $[\hat{g}]$ lifts to the first of these new classes.

(b) Suppose that g is not conjugate to zg in $2{}^{\cdot}\mathrm{Fi}_{22}$ and $[g]$ fuses with $[h]$ in $2{}^{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 11A$ and $\hat{h} \in 11B$, as in Figure 2(e)). It follows that $g \sim h$, $xg \sim xh$, $xzg \sim xzh$, and $zg \sim zh$ and these are representatives for the 4 new conjugacy classes *in that order*. Our Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{with} \quad k = 4 \text{ and } f_1 = f_2 = f_3 = f_4 = 1$$

In `cl4.gap` these matrices occur at indices 36, 41, and 47.

(c) Suppose that g is not conjugate to zg in $2{\cdot}\mathrm{Fi}_{22}$ and $[g]$ fuses with $[zh]$ in $2{\cdot}\mathrm{Fi}_{22}{:}2$ (*e.g.*, $\hat{g} \in 18A$ and $\hat{h} \in 18B$, as in Figure 2(f)). The 4 new conjugacy classes are those of $xg \sim xzh$, $g \sim zh$, $zg \sim h$, and $xzg \sim xh$ in that order. The Fischer matrix is

$$M([\hat{g}]) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \quad \text{with} \quad k = 4 \text{ and } f_1 = f_2 = f_3 = f_4 = 1$$

In `cl4.gap` these matrices occur at indices 51 and 56.

3. Suppose now that $\hat{g} \in \mathrm{Fi}_{22}{:}2 \setminus \mathrm{Fi}_{22}$. Then $g^x = xgx = xxzg = zg$ so g is always conjugate to zg in G.

(a) Suppose that $\hat{g}$ has order indivisible by 4 and lifts to one class in $2{\cdot}\mathrm{Fi}_{22}{:}2$ of elements of the same order, *e.g.*, $\hat{g} \in 2D$. Now, $(xg)(xg) = zg^2$ which must have even order, so $\hat{g}$ lifts to 2 classes in $\overline{G}$, the second (with representatives xg and xzg) consisting of elements of twice the order of those in the first which have the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(b) If $\hat{g}$ has order indivisible by 4 and lifts to one class in $2{\cdot}\mathrm{Fi}_{22}{:}2$ of elements of twice the order, *e.g.*, $\hat{g} \in 2F$. This means that $(g)(g) = zg^2$ so $\hat{g}$ lifts to 2 classes in $\overline{G}$, the first (with representatives g and zg) consisting of elements of twice the order of those in the second which have the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(c) If $\hat{g}$ has order indivisible by 4 and lifts to two classes in $2{\cdot}\mathrm{Fi}_{22}{:}2$ of elements of the same order, *e.g.*, $\hat{g} \in 6M$, then $(zg)(zg) = g^2$ and $(xg)(xg) = zg^2$. Therefore $\hat{g}$ lifts to 2 classes in $\overline{G}$, the second (with representatives xg and xzg) consisting of elements of twice the order of those in the first which have the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(d) Suppose that $\hat{g}$ has order divisible by 4 and lifts to 2 classes of elements of the same order as $\hat{g}$, *e.g.*, $\hat{g} \in 8F$. Then $\hat{g}$ lifts to two conjugacy classes in $2{\cdot}\mathrm{Fi}_{22}{:}2$, both consisting of elements of the same order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

(e) Suppose that $\hat{g}$ has order divisible by 4 and lifts to 2 classes of elements of the same order as $\hat{g}$. This case does not occur in Fi_{22}, an example is class $4A$ of A_5. In this case $\hat{g}$ lifts to two conjugacy classes in $2{\cdot}\mathrm{Fi}_{22}{:}2$, both consisting of elements of twice the order as $\hat{g} \in \mathrm{Fi}_{22}{:}2$.

In all of these cases the Fischer matrix is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ with $k = 2$ and $f_1 = f_2 = 1$.

(4.7.2) From Class Functions To Characters

Our Fischer matrices are only defined up to multiplication of rows by -1, and in the 4×4 matrices any permutation of the bottom 3 rows is possible. However, we have chosen class fusion from $2 \times 2{}^{\cdot}\mathrm{Fi}_{22}$ which forces us to use the matrices given above.

Using the Fischer matrices from above, and from GAP the character table of $\mathrm{Fi}_{22}{:}2$ and the projective character table of Fi_{22}, we assemble a table of class functions χ_i for $\overline{G}$ that obey row and column orthogonality. The element orders follow from the calculations above, as does class fusion from $2 \times 2{}^{\cdot}\mathrm{Fi}_{22}$. Because of our choice of ordering of the conjugacy classes we can also write down the projection map to $2{}^{\cdot}\mathrm{Fi}_{22}{:}2$

We use these maps to restrict each of our class functions to a class function ψ of the group $\mathrm{Fi}_{22}{:}2$. We then check that $\langle \psi, \chi \rangle \in \mathbb{N} \cup \{0\}$ for all $\chi \in \mathrm{Irr}\,(\mathrm{Fi}_{22}{:}2)$. (Whenever it was not, the reason was always because we had not used the correct Fischer matrix for our chosen ordering of the conjugacy classes.)

(4.7.3) Power Maps

To compute the power maps we observe that classes lying above $[\hat{g}]$ must power up to classes lying above $[\hat{g}^p]$ for all p, and for odd p elements in $[ng]$ must p-power to elements in $[ng^p]$ for all $n \in \langle x, z \rangle$. For $p = 2$ and $\hat{g} \in \mathrm{Fi}_{22}$ elements in $[ng]$ square to elements in $[g]$ and for outer elements the 2-power map is clear from item 3 of the case analysis.

We can also use GAP to compute possible power maps from the character table. Unusually, it produces unique p-power maps for our table and these agree with ours. Furthermore, these agree with the power maps of $2 \times 2{}^{\cdot}\mathrm{Fi}_{22}$.

To further test our class functions and power maps we check that all symmetric and anti-symmetric parts of all irreducibles have non-negative inner products with all irreducibles.

Finally, GAP produces four possible class fusions to Fi_{24}. There are two independent choices.

- There are two classes of elements of order 26 lying above class $13A$ of $\mathrm{Fi}_{22}{:}2$. These could fuse either way round to the algebraically conjugate classes $26B$ and $26C$ of Fi_{24}.

- Our labelling of the involutions x and xz was arbitrary, if we swap them then our choice of class representatives in cases 1(a)ii, 2(b), 2(c) is effected. For example in case 2(b) the class representatives would become (in order) $g \sim h$, $xzg \sim xzh$, $zg \sim zh$, and $zg \sim zh$. The labelling of x and xz corresponds to the other choice for class fusion. (The classes of elements of order 4 lying above $11A$ are not effected since they fuse to the same class in Fi_{24}.)

(4.8) A GAP Function For The General Case

We re-write `cl4.gap`, much more neatly and covering all possibilities for lifting/fusing classes from G to $2{}^{\cdot}G{:}2$. This is `ct22g2.gap` and works for arbitrary G.

The function

`CharacterTablleTwoSquaredGsplitTwo(t_g, t_g2, t_2g, t_2g2, proj1, proj2)`

assembles the character table. The arguments are:

`t_g`: The character table of G.

`t_g2`: The character table of $G{:}2$.

`t_2g`: The character table of $2{}^{\cdot}G$.

`t_2g2`: The character table of $2{}^{\cdot}G{:}2$.

`proj1`: The index in `ProjectivesInfo(t_g)` for the record with name $2{}^{\cdot}G$. This is usually 1.

`proj2`: The index in `ProjectivesInfo(t_g2)` for the record with name $2{}^{\cdot}G{:}2$. This is usually 1.

Notice throughout we have always assumed that in $2{}^{\cdot}G{:}2$ the normal subgroup $2{}^{\cdot}G$ has a complement. However, this need not always be the case: A_6 has three involutory automorphisms, σ, τ, and $\rho = \sigma\tau$ say, with

$$\mathrm{A}_6{:}\langle\sigma\rangle \cong \mathrm{S}_6 \qquad \mathrm{A}_6{:}\langle\tau\rangle \cong \mathrm{PGL}_2(9) \qquad \mathrm{A}_6{}^{\cdot}\langle\rho\rangle \cong \mathrm{M}_{10}$$

Our program can produce character tables for $2^2{}^{\cdot}\mathrm{S}_6$ and $2^2{}^{\cdot}\mathrm{PGL}_2(9)$, but not for $2^2{}^{\cdot}\mathrm{M}_{10}$. This is because there is no group $2{}^{\cdot}\mathrm{A}_6.2_3$, just a group $(4 \circ 2{}^{\cdot}\mathrm{A}_6){}^{\cdot}2_3$ which is isoclinic to $(2 \times 2{}^{\cdot}\mathrm{A}_6){}^{\cdot}2_3$. We can guess at a presentation for the latter,

$$
\begin{array}{lll}
\langle\, a, b, x, z, \rho \mid & z^2 = x^2 = [x,z] = [z,a] = [z,b] = [x,a] = [x,b] = 1 & \text{(the } O_2 \text{ subgroup)} \\
& a^2 = z, \quad b^4 = z, \quad (ab)^5 = 1 = (abb)^5 & \text{(lifted from } \mathrm{A}_6\text{)} \\
& [z,\rho] = 1, \quad [x,\rho] = z & \text{(action of } \rho \text{ on the } 2^2\text{)} \\
& a^\rho = z(babababb)^2, \quad \rho^2 = xb \qquad \rangle & \text{(action of } \rho \text{ on } \mathrm{A}_6\text{)}
\end{array}
$$

which we check in Magma. (This and some other presentations are on the CD at `/Fischer/pres.mag`.) Here a and b are pre-images of standard generators for A_6 and chosen so that $\rho^2 = b$ in $\mathrm{A}_6{}^{\cdot}2_3$. Once we are satisfied that we have the correct group we can compute its character table in Magma, this is shown in Figure 3. Figure 4 shows the character table of the isoclinic group $(4 \circ 2{}^{\cdot}\mathrm{A}_6){}^{\cdot}2_3$.

We re-ordered the classes and characters of $2^2{}^{\cdot}\mathrm{A}_6{}^{\cdot}2_3$ to match as closely as possible the format obtained from our Fischer matrices method. We have similarly sorted the classes and characters of $4{}^{\cdot}\mathrm{A}_6{}^{\cdot}2_3$ so that comparison of the character tables is easy. The usual relationship between character tables of isoclinic groups becomes apparent: Characters that faithfully represent the O_2 subgroup (rows 9–14) have values multiplied by i on the cosets of $\mathrm{A}_6{}^{\cdot}2_3$ containing x or xz. On the inner half of the group these are precisely the classes where the element orders differ.

	2	6	5	6	5	5	2	2	2	2	5	4	5	2	2	2	2	3	3	4	4	4	4
	3	2	2	2	0	0	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
		1a	2a	2b	4b	4a	3a	6a	6b	6c	8a	8b	8c	5a	10a	10b	10c	8d	8e	16a	16b	16c	16d
	2P	1a	1a	1a	2b	2b	3a	3a	3a	3a	4b	4b	4b	5a	5a	5a	5a	4a	4a	8a	8a	8c	8c
	3P	1a	2a	2b	4b	4a	1a	2a	2a	2b	8c	8b	8a	5a	10b	10a	10c	8e	8d	16d	16c	16b	16a
	5P	1a	2a	2b	4b	4a	3a	6a	6b	6c	8c	8b	8a	1a	2a	2a	2b	8d	8e	16c	16d	16a	16b
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1
X.12	+	10	10	10	2	2	1	1	1	1	-2	-2	-2	0	0	0	0	0	0	0	0	0	0
X.17	+	16	16	16	0	0	-2	-2	-2	-2	0	0	0	1	1	1	1	0	0	0	0	0	0
X.7	+	9	9	9	1	1	0	0	0	0	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1
X.8	+	9	9	9	1	1	0	0	0	0	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	-1
X.15	0	10	10	10	-2	-2	1	1	1	1	0	0	0	0	0	0	0	0	0	i2	-i2	-i2	i2
X.16	0	10	10	10	-2	-2	1	1	1	1	0	0	0	0	0	0	0	0	0	-i2	i2	i2	-i2
X.5	-	8	0	-8	0	0	-1	3	-3	1	0	0	0	-2	0	0	2	0	0	0	0	0	0
X.6	-	8	0	-8	0	0	-1	-3	3	1	0	0	0	-2	0	0	2	0	0	0	0	0	0
X.19	-	16	0	-16	0	0	-2	0	0	2	0	0	0	1	1+2b5	*	-1	0	0	0	0	0	0
X.20	-	16	0	-16	0	0	-2	0	0	2	0	0	0	1	*	1+2b5	-1	0	0	0	0	0	0
X.21	-	20	0	-20	0	0	2	0	0	-2	2r2	0	-2r2	0	0	0	0	0	0	0	0	0	0
X.22	-	20	0	-20	0	0	2	0	0	-2	-2r2	0	2r2	0	0	0	0	0	0	0	0	0	0
X.3	0	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	i	-i	i	-i	i	-i
X.4	0	1	-1	1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	-1	1	-i	i	-i	i	-i	i
X.11	+	10	-10	10	2	-2	1	-1	-1	1	2	-2	2	0	0	0	0	0	0	0	0	0	0
X.18	+	16	-16	16	0	0	-2	2	2	-2	0	0	0	1	-1	-1	1	0	0	0	0	0	0
X.9	0	9	-9	9	1	-1	0	0	0	0	-1	1	-1	-1	1	1	-1	i	-i	-i	i	-i	i
X.10	0	9	-9	9	1	-1	0	0	0	0	-1	1	-1	-1	1	1	-1	-i	i	i	-i	i	-i
X.13	+	10	-10	10	-2	2	1	-1	-1	1	0	0	0	0	0	0	0	0	0	r2	r2	-r2	-r2
X.14	+	10	-10	10	-2	2	1	-1	-1	1	0	0	0	0	0	0	0	0	0	-r2	-r2	r2	r2

Figure 3: The character table of $(2 \times 2^{\cdot}A_6)^{\cdot}2_3$. (Computed in Magma.)

2	6	5	6	5	5	2	2	2	2	5	4	5	2	2	2	2	3	3	4	4	4	4
3	2	2	2	0	0	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
	1a	4a	2a	4b	2b	3a	12a	12b	6a	8b	8a	8c	5a	20a	20b	10a	4c	4d	16a	16b	16c	16d
2P	1a	2a	1a	2a	1a	3a	6a	6a	3a	4b	4b	4b	5a	10a	10a	5a	2b	2b	8b	8b	8c	8c
3P	1a	4a	2a	4b	2b	1a	4a	4a	2a	8b	8a	8c	5a	20a	20b	10a	4d	4c	16b	16a	16d	16c
5P	1a	4a	2a	4b	2b	3a	12a	12b	6a	8c	8a	8b	1a	4a	4a	2a	4c	4d	16c	16d	16a	16b
X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1
X.3	10	10	10	2	2	1	1	1	1	−2	−2	−2	0	0	0	0	0	0	0	0	0	0
X.4	16	16	16	0	0	−2	−2	−2	−2	0	0	0	1	1	1	1	0	0	0	0	0	0
X.6	9	9	9	1	1	0	0	0	0	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1
X.5	9	9	9	1	1	0	0	0	0	1	1	1	−1	−1	−1	−1	1	1	−1	−1	−1	−1
X.7	10	10	10	−2	−2	1	1	1	1	0	0	0	0	0	0	0	0	0	i2	i2	−i2	−i2
X.8	10	10	10	−2	−2	1	1	1	1	0	0	0	0	0	0	0	0	0	−i2	−i2	i2	i2
X.17	8	0	−8	0	0	−1	−3i	3i	1	0	0	0	−2	0	0	2	0	0	0	0	0	0
X.18	8	0	−8	0	0	−1	3i	−3i	1	0	0	0	−2	0	0	2	0	0	0	0	0	0
X.19	16	0	−16	0	0	−2	0	0	2	0	0	0	1	−i5	i5	−1	0	0	0	0	0	0
X.20	16	0	−16	0	0	−2	0	0	2	0	0	0	1	i5	−i5	−1	0	0	0	0	0	0
X.21	20	0	−20	0	0	2	0	0	−2	2i2	0	−2i2	0	0	0	0	0	0	0	0	0	0
X.22	20	0	−20	0	0	2	0	0	−2	−2i2	0	2i2	0	0	0	0	0	0	0	0	0	0
X.9	1	−1	1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	−1	1	−i	i	−i	i	−i	i
X.10	1	−1	1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	−1	1	i	−i	i	−i	i	−i
X.11	10	−10	10	2	−2	1	−1	−1	1	2	−2	2	0	0	0	0	0	0	0	0	0	0
X.12	16	−16	16	0	0	−2	2	2	−2	0	0	0	1	−1	−1	1	0	0	0	0	0	0
X.13	9	−9	9	1	−1	0	0	0	0	−1	1	−1	−1	1	1	−1	−i	i	i	−i	i	−i
X.14	9	−9	9	1	−1	0	0	0	0	−1	1	−1	−1	1	1	−1	i	−i	−i	i	−i	i
X.15	10	−10	10	−2	2	1	−1	−1	1	0	0	0	0	0	0	0	0	0	−r2	r2	r2	−r2
X.16	10	−10	10	−2	2	1	−1	−1	1	0	0	0	0	0	0	0	0	0	r2	−r2	−r2	r2

Figure 4: The character table of $(4 \circ 2{\cdot}A_6){\cdot}2_3$. (Stored in GAP.)

Chapter 5

The Character Tables Of A Maximal Subgroup Of The Monster, And Some Related Groups

While finding conjugacy class representatives for $\mathbb{M}$ we wished to perform structure constant calculations in the maximal subgroup $N(3B) = 3^{1+12} {}^{\cdot} 2{}^{\cdot}\mathrm{Suz}{:}2$ of $\mathbb{M}$, but the character table of this group is not known. Calculation of the character table of the non-split extension of 3^{1+12} by $2{}^{\cdot}\mathrm{Suz}{:}2$ does not particularly lend itself to calculation by the technique of Fischer matrices. To further complicate matters, there are two (isomorphism classes of) groups of this shape and only one of them is a subgroup of $\mathbb{M}$. We overcome both the difficulties and subtleties by using a more round-about route in which we explicitly construct certain related groups and use the work of [29].

First we construct explicitly a representation of the split extension of 3^{12} by $6{}^{\cdot}\mathrm{Suz}{:}2$. We use this to determine the conjugacy classes as pairs $(g, \mathbf{v})$ for g a class representative in $6{}^{\cdot}\mathrm{Suz}{:}2$ and $\mathbf{v} \in 3^{12}$. This allows us to find some of the Fischer matrices for this extension. However, there are many ambiguities in the smaller matrices, and things are even worse for the larger matrices. The combined effect of all these ambiguities is to admit a massive number of possible character tables. We therefore prefer a more concrete approach.

Secondly, we construct a representation of $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$ and determine its conjugacy classes. We think of these as triples $(g, \mathbf{v}, \lambda)$ where g and $\mathbf{v}$ are as before and $\lambda \in \mathbb{F}_3$. This also allows us to compute the projection map to $N(3B) \leqslant \mathbb{M}$ and the power maps.

Corollary 6 tells us how to obtain the characters. The inertia groups are:

(i) 3^{1+12}:6·Suz:2. This case corresponds to the trivial character. Our description of the conjugacy classes makes it trivial to inflate the 174 irreducible characters of 6·Suz:2.

(ii) 3^{1+12}:6·Suz fixing the degree 729 characters of 3^{1+12}. Inducing characters from this subgroup is easy once we have determined the 729-dimensional characters and the class fusion from 6·Suz to 6·Suz:2. This is the subject of Section 5.3.

(iii) 3^{1+12}:$((3 \times U_5(2))$:2) fixes a linear character in an orbit of length 65520. Induction from this subgroup is the subject of Section 5.5

(iv) 3^{1+12}:$(3.3^5$:M_{11}:2) fixes a linear character in an orbit of length 465920.

At the time of writing we have found 463 out of 533 irreducible characters of 3^{1+12}:6·Suz:2. This comprises all of the characters from sources (i) and (ii), 55 characters from (iii), and 24 characters from (iv).

Most of the computations are performed in Magma, but we switch to GAP to assemble the character table. This lets use use the character tables stored in GAP, and the power map construction functions. The data we need to exchange between Magma and GAP consists mostly of arrays and so can be stored in a text file that can be read by both systems. (Most data is included on the CD, and is referred to throughout this chapter.)

(5.1) The Conjugacy Classes Of 6·Suz:2

Before attempting any downward extension at all we must correctly identify the conjugacy classes of 6·Suz:2. This will make our work consistent with, and allow us to reliably use, existing information and results, *e.g.,* the character tables stored in GAP.

In Section 5.3 we will also require the class fusion from 6·Suz and identifying the conjugacy classes of this group is slightly harder (there are many pairs of algebraically conjugate classes). The web Atlas [52] gives words in the standard generators for the conjugacy classes of Suz up to algebraic conjugacy, but we must identify *all* the conjugacy classes of 6·Suz and 6·Suz:2.

We begin in Magma with the 24-dimensional representation of 6·Suz:2 over $\mathbb{F}_3$ and use the web Atlas words to find a subgroup 6·Suz. Magma can produce a list of the conjugacy classes of both of these groups, and the class fusion. The outer classes of 6·Suz:2 are easy to identify, so the problem is reduced to identifying the conjugacy classes of 6·Suz.

We use words in standard generators to switch freely between representations: Standard generators for 6·Suz are A and B where $A^4 = 1 = B^3$ and $|AB| = 13$. We use the notation $\mathbf{di}_p$ to refer to the i-th representation of (a cover of) Suz of dimension d in characteristic p with character printed in the ATLAS or Brauer tables [24]. The ordering is as in the character tables, and $i \in \{a, b, c, \dots\}$.

Identifying conjugacy classes is not straightforward for the following reasons. Let X and X' be algebraically conjugate conjugacy classes containing elements x and x' respectively. Let Y and Y' be another such pair with class representatives y and y'. Suppose that there exists a character ϕ, either ordinary or Brauer, such that $\phi(x) \neq \phi(x')$ and $\phi(y) \neq \phi(y')$. Then if we have already identified X and X' then we must identify Y and Y' to be consistent with ϕ. We shall say that (X, X') is *connected* to (Y, Y') (via ϕ).

	$\dots$	X	X'	$\dots$	Y	Y'	$\dots$
ϕ	$\dots$	α	β	$\dots$	γ	δ	$\dots$
ϕ'	$\dots$	β	α	$\dots$	δ	γ	$\dots$

This defines an equivalence relation on the pairs of algebraically conjugate conjugacy classes. In each equivalence class we can make just one choice for the class names, the rest being determined by the connections. Furthermore, there will usually be another character ϕ' related to ϕ as shown in the table above. Representations affording these characters may exist on [52], in which case we must be consistent with these too.

From what is known of the Brauer character tables of Suz and its covers [24] we find that all pairs of algebraically conjugate classes are connected, so we have just one choice: With standard generators a and b for Suz we choose $ab \in 13A$ (rather than $13B$).

It is straightforward to identify most of the conjugacy classes of 6·Suz from the element orders, centraliser orders, and complex character values computed from $\mathbf{12a}_0$ [5, 31]. We can be sure that we have the correct such representation (the one with character values printed in the ATLAS) by using a word for the central element "$-\omega$": In the complex representation it evaluates to the scalar matrix with $-\exp(2\pi i/3)$ on the diagonal. The representation $\mathbf{12a}_2$ is over $\mathbb{F}_4$ and in MeatAxe notation we have $\mathbb{F}_4 = \{0, 1, 2, 3\}$ with 2 corresponding to ω. Therefore in this representation our word must evaluate to the scalar matrix with 2 on the diagonal.

The conjugacy classes of 6·Suz we cannot identify by this means are those lying above classes $12D$ or $24A$ of Suz (in each case there are two). Both of these classes lift to classes which cannot be distinguished by element order, character value, or by using the power maps. The smallest dimension in which we can distinguish these classes is 1892 over $\mathbb{F}_{25}$ (a representation of $2^{\cdot}$Suz). This representation is not available on the web Atlas so we must construct it:

- From the 5-modular decomposition matrix for Suz [31] we find that the representation we require is a constituent of the 5-modular decomposition of $\mathbf{4928a}_0$.
- Using the GAP character table we find that $\mathbf{12a}_0 \otimes \mathbf{429a}_0$ splits as $\mathbf{220}_0 + \mathbf{4928a}_0$.
- To obtain $\mathbf{429a}_0$ we find that $\mathbf{12a}_0^{\otimes 3}$ contains $\mathbf{220}_0$ and in turn $\mathbf{220}_0 \otimes \mathbf{12a}_0$ contains $\mathbf{429a}_0$.

Reducing modulo 5 we have $\mathbf{429a}_5 = \mathbf{363b}_5 + \mathbf{66a}_5$ so $\mathbf{1892a}_5$ is a constituent of $\mathbf{363b}_5 \otimes \mathbf{12a}_5$. As $\mathbf{12a}_5$ is available on the web Atlas we can now construct $\mathbf{1892a}_5$ and identify the remaining conjugacy classes.

The representations used are on the CD at `/character/classes`.

(5.1.1) A Representation Of 3^{12}:$6^{\cdot}$Suz:2

Let $V \cong 3^{12}$. Two matrices, C_2 and D_2 say, that generate $2^{\cdot}$Suz:2 represented on V are available on the web Atlas [52]. There is a 24-dimensional representation of $6^{\cdot}$Suz:2 over $\mathbb{F}_3$, also available on the web Atlas [52], with (standard) generators C_6 and D_6. Hence via the projection map

$$6^{\cdot}\mathrm{Suz{:}2} \to 2^{\cdot}\mathrm{Suz{:}2} \quad \text{given by} \quad C_6 \mapsto C_2 \text{ and } D_6 \mapsto D_2$$

and the representation of $2^{\cdot}$Suz:2 on V we obtain an un-faithful representation of $6^{\cdot}$Suz:2 on V.

Let g be a word in two generators. Let $f_6(g)$ be g evaluated in the 24-dimensional representation of $6^{\cdot}$Suz:2 and let $f_2(g)$ be the 12×12 matrix representing the projection down to $2^{\cdot}$Suz:2. Then

$$V{:}6^{\cdot}\mathrm{Suz{:}2} \cong \left\{ \begin{pmatrix} f_6(g) & 0 & 0 \\ 0 & f_2(g) & 0 \\ 0 & \mathbf{v} & 1 \end{pmatrix} \mid g \text{ is a word for an element of } 6^{\cdot}\mathrm{Suz{:}2} \text{ and } \mathbf{v} \in V \right\}$$

In practice we think of such an element as a pair $(g, \mathbf{v})$ where multiplication is given by $(g, \mathbf{v})(h, \mathbf{w}) = (gh, \mathbf{v}h + \mathbf{w})$ and $\mathbf{v}g$ means the image of $\mathbf{v}$ under the action of $f_2(g)$. We determine the conjugacy classes of this group.

A conjugacy class $[f_6(g_i)]$ of $6^{\cdot}$Suz:2 lifts to a number of conjugacy classes, say $[(g_i, \mathbf{v}_{i_j})]$, of V:$6^{\cdot}$Suz:2 and following [35] the $\mathbf{v}_{i_j}$ are determined in 2 steps.

1. V acts by conjugation on the coset $f_2(g_i)V$ by $\mathbf{w}$: $(g_i, \mathbf{v}) \mapsto (g_i, -\mathbf{w}g_i + \mathbf{v} + \mathbf{w})$.

2. These orbits are now fused by the action of the centraliser of $f_6(g_i)$. We must compute the centraliser in 6·Suz:2 of $f_6(g_i)$ and use its image in 2·Suz:2 to determine the fusions: The pre-image in 6·Suz:2 of $C_{2\cdot\mathrm{Suz:2}}(f_2(g_i))$ is not always equal to $C_{6\cdot\mathrm{Suz:2}}(f_6(g_i))$.

All these calculations can be performed in Magma [47] and the programs and results are on the CD at `/character/6s2/`. Firstly we use the `ConjugacyClasses(G)` function to find the conjugacy classes of 6·Suz:2 in the 24-dimensional representation. We compute the class fusion from 6·Suz and thus identify all of the inner classes. The outer classes are straightforward to identify.

Having calculated the representative vectors for the classes lying above $[f_6(g_i)]$ we store these in an array for each i, and we think of the class representatives as pairs $(g_i, \mathbf{v}_{i_j})$ which we also refer to by the label (i, j).

(5.1.2) A Representation Of 3^{1+12}:6·Suz:2

We construct a 38-dimensional representation of 3^{1+12}:6·Suz:2 similarly to above. Most of the details can be found in [29], but we repeat some of the calculations since we found it necessary to re-order the co-ordinates of the module to achieve computational benefits. Programs and results are on the CD and are described in Appendix A.4.

Elements of 3^{1+12}:6·Suz:2 can be thought of as triples $(g, \mathbf{v}, \lambda)$ where g and $\mathbf{v}$ are as in the previous section and λ is in the new normal subgroup. Multiplication of such triples is rather unwieldy—*cf.* equation (19) on page 58—so 38-dimensional matrices over $\mathbb{F}_3$ are more convenient.

For convenience write g_2 for $f_2(g)$. We find a matrix S^{-1} corresponding to a symplectic form on V that is invariant under 2·Suz but elements g_2 in 2·Suz:2 outside of this subgroup invert (*i.e.,* negate) the form. Equivalently, S^{-1} conjugates g_2 to its inverse transpose or to $-g_2^{-\top}$ (respectively) and so can readily be found using Standard Basis in the MeatAxe [27]. For an outer element g_2 this gives

$$\begin{pmatrix} g_2 & 0 \\ \mathbf{v} & 1 \end{pmatrix}^{-\top} = \begin{pmatrix} g_2^{-\top} & -g_2^{-1}\mathbf{v}^\top \\ 0 & 1 \end{pmatrix}$$

$$\text{so} \quad \begin{pmatrix} S^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g^{-\top} & -g_2^{-1}\mathbf{v}^\top \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} g_2 & -g_2 S^{-1}\mathbf{v}^\top \\ 0 & -1 \end{pmatrix}$$

and so we can paste together matrices

$$\begin{pmatrix} 1 & \mathbf{v} & \lambda \\ 0 & g_2 & -g_2 S^{-1}\mathbf{v}^\top \\ 0 & 0 & -1 \end{pmatrix} \tag{17}$$

corresponding to triples $(g, \mathbf{v}, \lambda)$. If g_2 is inner the only difference is that the bottom right entry must be 1 instead of -1. The 38-dimensional representation of 3^{1+12}:6·Suz:2 is obtained by taking a sub-direct product with 6·Suz:2, so the matrices are of the form

$$\begin{pmatrix} g_6 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{v} & \lambda \\ 0 & 0 & g_2 & -g_2 S^{-1}\mathbf{v}^\top \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{18}$$

where $g_6 = f_6(g)$. It is convenient to use the label (i, j, λ) to refer to the element of 3^{1+12}:6·Suz:2 obtained by pasting together $f_6(g_i)$, $f_2(g_i)$, $\mathbf{v}_{i_j}$, and λ as in equation 18. Standard generators for this group are defined in [29]. C and D are standard generators (up to automorphism) for the subgroup 6·Suz:2 that complements the 3^{1+12}, *i.e.*,

$$C = \begin{pmatrix} C_6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} D_6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The outer automorphism of 6·Suz:2 multiplies elements in the outer half of the group by the central involution, and so we may need to replace C with its inverse. We solve this problem at the end of Section 5.3.4.

The third generator, E, is an element of 3^{1+12}. It is inverted by the central involution of 2·Suz and centralised by a certain subgroup $U_5(2)$. To find E we use words from the web Atlas [52] to find standard generators for 2·Suz in 2·Suz:2 and then use more such words to find generators u_1 and u_2 for the subgroup $2 \times U_5(2)$. We find that u_1^5 is an involution that commutes with u_1 and u_2, therefore u_1^5 is the central involution. The order of u_2 is even and $u_1 u_2$ has odd order, so u_2 is not in the complementary

$U_5(2)$. Therefore, using the MeatAxe, we find a vector $\mathbf{e}$ with $u_1\mathbf{e} = u_2\mathbf{e} = -\mathbf{e}$ and paste it into the matrix

$$E = \begin{pmatrix} I_{24} & 0 & 0 & 0 \\ 0 & 1 & \mathbf{e} & 0 \\ 0 & 0 & I_{12} & -S^{-1}\mathbf{e}^\top \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5.1.3) Conjugacy Classes

To determine the conjugacy classes of $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$ we need to determine conjugacy amongst the elements labelled $(i, j, 0)$, $(i, j, 1)$ and $(i, j, 2)$ for all pairs (i, j) corresponding to conjugacy classes of $3^{12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$. Some of this work can be done directly in Magma, but there are a few computationally hard cases that require further investigation.

Let g and h be words for which $g_6 = f_6(g)$ and $h_6 = f_6(h)$ lie in the inner half of $6{}^{\cdot}\mathrm{Suz}{:}2$. Clearly $g_2 = f_2(g)$, and $h_2 = f_2(h)$ lie in the inner half of $2{}^{\cdot}\mathrm{Suz}{:}2$, and the 38 dimensional matrices known as $(g, \mathbf{v}, \lambda)$ and $(h, \mathbf{w}, \mu)$ will lie in the inner half of $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$. Conjugating $(g, \mathbf{v}, \lambda)$ by $(h, \mathbf{w}, \mu)$ we find that

$$\begin{pmatrix} h_6^{-1} & 0 & 0 & 0 \\ 0 & 1 & -\mathbf{w}h_2^{-1} & -\mu - \mathbf{w}S^{-1}\mathbf{w}^\top \\ 0 & 0 & h_2^{-1} & S^{-1}\mathbf{w}^\top \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_6 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{v} & \lambda \\ 0 & 0 & g & -g_2S^{-1}\mathbf{v}^\top \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_6 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{w} & \mu \\ 0 & 0 & h_2 & -h_2S^{-1}\mathbf{w}^\top \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} h_6^{-1}g_6h_6 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{w} + \mathbf{v}h_2 - \mathbf{w}h_2^{-1}g_2h_2 & \lambda - \mathbf{v}h_2S^{-1}\mathbf{w}^\top + \mathbf{w}h_2^{-1}g_2h_2S^{-1}\mathbf{w}^\top + \mathbf{w}h_2^{-1}g_2S^{-1}\mathbf{v}^\top - \mathbf{w}S^{-1}\mathbf{w} \\ 0 & 0 & h_2^{-1}g_2h_2 & -h_2^{-1}g_2h_2S^{-1}\mathbf{w}^\top - h_2^{-1}g_2S^{-1}\mathbf{v}^\top + S^{-1}\mathbf{w}\top \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

so that

$$\lambda \longmapsto \lambda - \mathbf{v}h_2S^{-1}\mathbf{w}^\top + \mathbf{w}h_2^{-1}g_2h_2S^{-1}\mathbf{w}^\top + \mathbf{w}h_2^{-1}g_2S^{-1}\mathbf{v}^\top - \mathbf{w}S^{-1}\mathbf{w} \tag{20}$$

Similarly, if h_6 and h_2 are outer then λ is transformed as follows:

$$\lambda \longmapsto -\lambda - \mathbf{v}h_2S^{-1}\mathbf{w}^\top + \mathbf{w}h_2^{-1}g_2h_2S^{-1}\mathbf{w}^\top - \mathbf{w}h_2^{-1}g_2S^{-1}\mathbf{v}^\top + \mathbf{w}S^{-1}\mathbf{w} \tag{21}$$

Suppose that $(h, \mathbf{w}, \mu)$ centralises $(g, \mathbf{v}, \lambda)$. Then we require h_6 to centralise g_6 (if so then it follows that h_2 centralises g_2), and from the $(2, 3)$ entry of the matrix on the right hand side of equation (19) we obtain the condition $\mathbf{v} - \mathbf{v}h_2 = \mathbf{w} - \mathbf{w}g_2$. We can use this condition as follows: Let C be the image in 2·Suz:2 of the centraliser in 6·Suz:2 of g_6 (so $h_2 \in C$) and let Q be the stabiliser in C of $\mathbf{v}$. Let T be a transversal for Q in C. The dichotomy between h_2 being inner and outer forces us to use both an inner and outer representative for each coset. If C contains an outer element q then we let $T' = \{t, qt \mid t \in T\}$ We now need only examine pairs $(h_2, \mathbf{w})$ for $h_2 \in T'$ and $\mathbf{w}$ in the set

$$\{\mathbf{w} \in V \mid \mathbf{w} - \mathbf{w}g_2 = \mathbf{v} - \mathbf{v}h_2\}$$

For each of these pairs $(h_2, \mathbf{w})$ we calculate the value of

$$-\mathbf{v}h_2S^{-1}\mathbf{w}^\top + \mathbf{w}g_2S^{-1}\mathbf{w}^\top + \mathbf{w}h_2^{-1}g_2S^{-1}\mathbf{v}^\top - \mathbf{w}S^{-1}\mathbf{w} \tag{22}$$

(h_2 inner, corresponding to equation (20)), or

$$-\mathbf{v}h_2S^{-1}\mathbf{w}^\top + \mathbf{w}g_2S^{-1}\mathbf{w}^\top - \mathbf{w}h_2^{-1}g_2S^{-1}\mathbf{v}^\top + \mathbf{w}S^{-1}\mathbf{w} \tag{23}$$

(h_2 outer, corresponding to equation (21)) which we use in the following case analysis. We also compute a fingerprint $L(g, \mathbf{v}, \lambda)$ which we use to prove non-fusion: Let X and Y be n-dimensional matrices with coefficients in a field $\mathbb{F}$. Let w be a polynomial in one variable, so $w(X)$ is an element of the algebra of matrices generated by X. If $\exists Z \in \mathrm{GL}_n(\mathbb{F})$ with $X^Z = Y$ then Z also conjugates $w(X)$ to $w(Y)$ and therefore $w(X)$ and $w(Y)$ have the same rank (*cf.* Section 2.1.1). To use this, we store a list of randomly chosen polynomials $w_1, w_2, \ldots, w_m$ and given a matrix X we produce a list $L(X) = (\operatorname{rk} w_1(X), \ldots, \operatorname{rk} w_m(X))$ of integers. Hence if $L(X) \neq L(Y)$ then X is not conjugate to Y.

1. If $L(g, \mathbf{v}, 0)$, $L(g, \mathbf{v}, 1)$, and $L(g, \mathbf{v}, 2)$ are distinct then there can be no fusion.

2. If two of the fingerprints are distinct and h is outer then the classes with equal fingerprints fuse.

3. If two of the fingerprints are distinct and g is not centralised by any outer element then no fusion occurs.
4. Otherwise proceed as described to find non-zero values for formula (22) or formula (23).
   (a) If h is inner and formula (22) is non-zero then all 3 classes fuse.
   (b) If h is outer then a value in formula (23) of $\begin{array}{c}0\\1\\2\end{array}$ fuses classes $\begin{array}{c}1\ \&\ 2\\0\ \&\ 1\\0\ \&\ 2\end{array}$.
   Note that it is not necessarily the case that a value of 0 occurs as we have not always chosen the zero vector as the representative of its orbit. If any 2 of these values occur then all 3 classes fuse.

A program for performing these calculations is on the CD at
`/character/V6s2/pasting39case.mag`.

(5.2) The Quotient $3^{1+12}\cdot 2\cdot$Suz:2 and Power Maps

In the previous section we determined the conjugacy classes of 3^{1+12}:6·Suz:2. In this section we shall determine the quotient map to $3^{1+12}\cdot 2\cdot$Suz:2, the class fusion from this quotient to the Monster, and the power maps.

(5.2.1) The Quotient And Fusion To The Monster

Using our computer programs in Chapter 2 and our identification of the conjugacy classes of $\mathbb{M}$ in Chapter 3 we can determine the conjugacy class in $\mathbb{M}$ of any element of $3^{1+12}\cdot 2\cdot$Suz:2, up to algebraic conjugacy and except for elements of order 27.

Projection from 3^{1+12}:6·Suz:2 to $3^{1+12}\cdot 2\cdot$Suz:2 is trivial since we made sure that our generators C, D, E for the latter group are pre-images of those by the same names for the former group, as described in [29]. This paper also furnished us with a word for an element, q say, in the 3-group we must quotient out.

Once Magma has created a base and strong generating set for our 38-dimensional $\mathbb{F}_3$-representation of $\langle C, D, E\rangle = 3^{1+12}$:6·Suz:2 it can quickly produce a word in these generators for a given element. Using our results from the previous section we paste together—as in equation (18)—a conjugacy class representative and then obtain a word for it. These words are used as input to the programs `class2.c` and `class7.c` described in Section 2.3.3. We then calculate the conjugacy class in $\mathbb{M}$ to which the image in $3^{1+12}\cdot 2\cdot\mathrm{Suz}{:}2$ of this element fuses.

We wish to reverse the process in Section 4.2 in order to determine the conjugacy class representatives for $3^{1+12}\cdot 2\cdot\mathrm{Suz}{:}2$. If $z \in 3^{1+12}{:}6\cdot\mathrm{Suz}{:}2$ then we may take the coset $z\langle q\rangle$ as a conjugacy class representative for $3^{1+12}\cdot 2\cdot\mathrm{Suz}{:}2$. It is therefore necessary to determine the conjugacy classes in $3^{1+12}{:}6\cdot\mathrm{Suz}{:}2$ of zq and zq^2.

We begin with a list L of all triples of integers (i, j, k) that label the conjugacy classes of $3^{1+12}{:}6\cdot\mathrm{Suz}{:}2$, as determined in Section 5.1.3. We determine the conjugacy class labels (i', j', k') of zq and (i'', j'', k'') of zq^2. (These classes need not be distinct.) Any of these labels different from (i, j, k) we remove from L and we continue. On termination L is a list of labels of conjugacy classes of $3^{1+12}{:}6\cdot\mathrm{Suz}{:}2$ that are in bijective correspondence with the conjugacy classes of $3^{1+12}\cdot 2\cdot\mathrm{Suz}{:}2$.

To determine the label of the conjugacy class of zq we begin with a list K of all class labels which we first reduce to a list of possible labels for the class of zq. To do this we remove those elements of K for which the class representative k does not meet the following criteria:

(i) The images in $3^{1+12}\cdot 2\cdot\mathrm{Suz}{:}2$ of k and zq fuse to the same conjugacy class in $\mathbb{M}$.

(ii) zq and k have the same fingerprint.

This considerably reduces the size of K: Around 44% of the time $|K| = 1$, otherwise $|K| = 2$ is typical and the worst case is $|K| = 6$. We use the `IsConjugate` function in Magma to determine which class in K contains zq. However, since a `false` result takes considerably longer than `true` we treat `IsConjugate` as a *Las Vegas* algorithm—we test all possibilities at once on our different machines, the one that returns `true` finishes first.

This process must be repeated for zq^2.

(5.2.2) Power Maps

We use the same technique to find the power maps for $3^{1+12}{:}6\cdot\mathrm{Suz}{:}2$. Let z be a class representative and $p \in \{2, 3, 5, 7, 11, 13\}$. For each p we wish to determine the conjugacy class of z^p. The following information allows us to produce an approximate p-th power map *i.e.,* a list of possible conjugacy classes for z^p.

(i) Power maps lifted from $6{}^{\cdot}\mathrm{Suz}{:}2$.

(ii) Power maps of $\mathbb{M}$ restricted to $N(3B)$ and lifted using the projection map calculated above.

(iii) Using characters produced from the above two sources, we can use the power map construction functions in GAP to further refine the approximation.

(iv) Fingerprints.

This usually leaves just 1 or 2 possibilities for the conjugacy class of z^p. Testing these with Magma is feasible, and we reduce the possibilities far enough so that GAP produces a unique power map.

(5.2.3) The First 174 Characters

Now that we have the conjugacy classes, power maps, and the projection map, all that remains is to find the characters! There are 174 irreducible characters of $6{}^{\cdot}\mathrm{Suz}{:}2$ which we inflate to $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$.

(5.3) The 729-dimensional representations of $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$

It turns out that the two irreducible representations of 3^{1+12} of degree 729 extend to $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}$, and are fused to a 1458-dimensional representation in $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$ (the 3-part of the centre of $6{}^{\cdot}\mathrm{Suz}$ is represented trivially). We shall construct all of these representations over various fields for the following purposes:

(i) Construction over $\mathbb{F}_{31}$ allows us to lift eigenvalues to $\mathbb{C}$ and so find values of the degree 729 characters.

(ii) Construction over $\mathbb{F}_{103}$ and $\mathbb{F}_{223}$ allows us to deduce the values of the degree 1458 character (which is rational).

Our generators will be images of C, D, and E; the standard generators of $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$. We will thus be able to switch freely between representations by using words (or straight line programs) in the generators.

Firstly, as in [29], we shall construct 3^{1+12} in 729 dimensions over our chosen field $\mathbb{F}$.

(5.3.1) A 729-dimensional module for 3^{1+12}

Let the basis vectors $\mathbf{e_v}$ of the 729-dimensional module be labelled by the vectors $\mathbf{v} \in \mathbb{F}_3^6$. The dual space of $\mathbb{F}_3^6$ consists of the linear maps $\chi_\mathbf{w}$ defined by $\chi_\mathbf{w}(\mathbf{v}) = \mathbf{v} \cdot \mathbf{w}$ (the usual inner product). Now let

$$\mathcal{T} = \{T_\mathbf{w} \colon \mathbf{e_v} \mapsto \mathbf{e_{v+w}}\}$$
$$\mathcal{D} = \{D_\mathbf{w} \colon \mathbf{e_v} \mapsto \omega^{\mathbf{v}\cdot\mathbf{w}}\mathbf{e_v}\}$$

(each map extended linearly to $\mathbb{F}^{729}$) where ω is a cube root of unity in $\mathbb{F}$. (We abuse notation by exponentiating ω by an element of $\mathbb{F}_3$.) Clearly $\mathcal{T}$ is isomorphic to the additive group of $\mathbb{F}_3^6$ and (recall we are extending linearly),

$$D_\mathbf{x}D_\mathbf{y}(\mathbf{e_z}) = D_\mathbf{x}(\omega^{\mathbf{y}\cdot\mathbf{z}}\mathbf{e_z}) = \omega^{\mathbf{y}\cdot\mathbf{z}}\omega^{\mathbf{x}\cdot\mathbf{z}}\mathbf{e_z} = \omega^{(\mathbf{x}+\mathbf{y})\cdot\mathbf{z}}(\mathbf{e_z}) = D_{\mathbf{x}+\mathbf{y}}(\mathbf{e_z})$$

so $\mathcal{D} \cong 3^6$ also. Similar calculations show that $D_\mathbf{y}T_\mathbf{x} = \omega^{\mathbf{x}\cdot\mathbf{y}}T_\mathbf{x}D_\mathbf{y}$. In particular, if $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_6\}$ is a basis for $\mathbb{F}_3^6$ then $[D_{\mathbf{x}_i}, T_{\mathbf{x}_j}] = \omega^{\delta_{i,j}}$. Hence $\langle D_\mathbf{x}, T_\mathbf{x}\rangle/\langle\omega\rangle \cong 3^{12}$ and the commutator map on $\langle D_\mathbf{x}, T_\mathbf{x}\rangle$ defines a symplectic form on this quotient vector space, *i.e.*, $\langle D_\mathbf{x}, T_\mathbf{x} \mid \mathbf{x} \in \mathbb{F}_3^6\rangle \cong 3^{1+12}$.

Clearly the $T_\mathbf{w}$ will be permutation matrices and the $D_\mathbf{w}$ will be diagonal. In order to write down some matrices we first order and label the elements of $\mathbb{F}_3^6$ as

$1 = 000000$					
$2 = 100000$	$3 = 200000$				
$4 = 010000$	$5 = 110000$	$6 = 210000$	$7 = 020000$	$8 = 120000$	$9 = 220000$
$10 = 001000$	$11 = 101000$	$12 = 201000$	$13 = 011000$	$14 = 111000$	$15 = 211000$
	$16 = 021000$	$17 = 121000$	$18 = 221000$	$19 = 002000$	$\dots$

from which it is clear that

$$T_{100000} = (1\,2\,3)(4\,5\,6)(7\,8\,9)\dots \qquad D_{100000} = \mathrm{diag}\,(1, \omega, \overline{\omega}, 1, \omega, \overline{\omega}, \dots)$$
$$T_{010000} = (1\,4\,7)(2\,5\,8)(3\,6\,9)\dots \qquad D_{010000} = \mathrm{diag}\,(1, 1, 1, \omega, \omega, \omega, \overline{\omega}, \overline{\omega}, \overline{\omega}, \dots)$$
$$T_{\mathbf{x}_i} = (j, j+3^i, j+2.3^i)\dots \qquad D_{\mathbf{x}_i} = \mathrm{diag}\,(\underbrace{1^{3^{i-1}}, \omega^{3^{i-1}}, \overline{\omega}^{3^{i-1}}}_{3^{6-i}\text{ times}})$$

We construct such matrices in GAP and write them out in MeatAxe format (`/character/729/esgGens/esg.gap`). At this point it is necessary to choose $\mathbb{F}$—we chose $\mathbb{F} = \mathbb{F}_{103}$ in which we use $\omega = 46$ and $\overline{\omega} = 56$.

(5.3.2) Extending To 3^{1+12}:2·Suz

There is an action of 2·Suz on 3^{12} which we wish to identify with our 12-dimensional $\mathbb{F}_3$ space consisting of the direct sum of $\mathbb{F}_3^6$ and its dual. In the previous section we described a symplectic basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_6, \chi_{\mathbf{x}_1}, \ldots, \chi_{\mathbf{x}_6}\}$ for this space.

As in Section 5.1.2 we find a matrix S^{-1} corresponding to a symplectic form that is invariant under the action of 2·Suz on 3^{12} (`/character/729/do.sh`). We now wish to find a symplectic basis so that the symplectic form becomes $\left(\begin{smallmatrix} 0 & -I_6 \\ I_6 & 0 \end{smallmatrix}\right)$ so that we can identify the co-ordinates with those of the previous section. Now,

$$S^{-1} = \begin{pmatrix} 0&0&0&0&0&0&2&1&1&1&2&1 \\ 0&0&0&0&0&0&1&1&2&1&2&2 \\ 0&0&0&0&0&1&0&1&2&1&0&2 \\ 0&0&0&0&1&0&1&0&0&2&1&2 \\ 0&0&0&2&0&0&0&2&1&0&1&0 \\ 0&0&2&0&0&0&2&0&1&0&2&0 \\ 1&2&0&2&0&1&0&0&2&0&0&1 \\ 2&2&2&0&1&0&0&0&0&1&1&0 \\ 2&1&1&0&2&2&1&0&0&0&0&1 \\ 2&2&2&1&0&0&0&2&0&0&2&0 \\ 1&1&0&2&2&1&0&2&0&1&0&0 \\ 2&1&1&1&0&0&2&0&2&0&0&0 \end{pmatrix}$$

and since the (i, j) entry is 0 for $1 \leqslant i, j \leqslant 4$ we can take the first 4 basis elements $\mathbf{t}_i$ to have a 1 in the ith position and 0 elsewhere. Next, $\mathbf{t}_5$ is in the null space of these 4 vectors, so if B is the matrix with rows $\mathbf{t}_1, \ldots, \mathbf{t}_4$ then $\mathbf{t}_5$ can be selected from the basis of the null space of BS^{-1}, computed by the MeatAxe. The next vector, $\mathbf{t}_6$, follows similarly. The final 6 basis elements $\mathbf{d}_i$ must have $\mathbf{d}_i S^{-1} \mathbf{t}_j^\top = \delta_{ij}$ and $\mathbf{d}_i S^{-1} \mathbf{d}_j^\top = 0$ for $j < i$ so now let B_i be the matrix with rows $\{\mathbf{t}_j \mid j \neq i\} \cup \{\mathbf{d}_1, \ldots, \mathbf{d}_i\}$. Then $B_i S^{-1} \mathbf{d}_{i+1}^\top = 0$ and so we can choose $\mathbf{d}_{i+1}$ from the basis of the null space of $B_i S^{-1}$. If B is the matrix with rows all 12 of the new basis vectors, then $BS^{-1}B^\top$ is the matrix of the symplectic form with respect to the new basis, and it is equal to $\left(\begin{smallmatrix} 0 & -I_6 \\ I_6 & 0 \end{smallmatrix}\right)$. Conjugating the generators for 2·Suz by B^{-1} changes basis to make them preserve this form.

We now need to find matrices in $\mathrm{GL}_{729}(\mathbb{F})$ that act by conjugation on $\{T_{\mathbf{w}_i}, D_{\mathbf{w}_i} \mid 1 \leqslant i \leqslant 6\}$ in the same way that 2·Suz acts on $\{\mathbf{t}_i, \mathbf{d}_i \mid 1 \leqslant i \leqslant 6\}$. Let $g \in$ 2·Suz and

$$\text{write } \mathbf{t}_i g = \sum_{j=1}^{6} \lambda_{ij}\mathbf{t}_j + \mu_{ij}\mathbf{d}_j \text{ to define } \overline{T}_i = \prod_{j=1}^{6} T_{\mathbf{w}_j}^{\lambda_{ij}} \prod_{j=1}^{6} D_{\mathbf{w}_j}^{\mu_{ij}}$$
$$\text{and write } \mathbf{d}_i g = \sum_{j=1}^{6} \nu_{ij}\mathbf{t}_j + \xi_{ij}\mathbf{d}_j \text{ to define } \overline{D}_i = \prod_{j=1}^{6} T_{\mathbf{w}_j}^{\nu_{ij}} \prod_{j=1}^{6} D_{\mathbf{w}_j}^{\xi_{ij}}$$

so $\overline{T}_i$ and $\overline{D}_i$ are, up to multiplication by scalars, the images of T_i and D_i under this action. We obtain a matrix $G \in \mathrm{GL}_{729}(\mathbb{F})$ that conjugates $T_{\mathbf{w}_i}$ to $\overline{T}_i$ and $D_{\mathbf{w}_i}$ to $\overline{D}_i$ by using the standard basis algorithm, which was briefly explained in Section 2.1.1. We take for the seed vector, $\mathbf{v}_1$, a non-zero vector in the 1-space fixed by $\langle D_{\mathbf{w}_i}\rangle$ and find the matrix M_1^{-1} which conjugates the $D_{\mathbf{w}_i}$ to standard basis. We obtain a corresponding base change matrix M_2^{-1} for the $\overline{D}_i$ (`/character/729/makeprod.pl` and `/character/729/do2.sh`). Thus $G = M_1^{-1}M_2$ conjugates $D_{\mathbf{w_i}}$ to $\overline{D}_i$ for all i. However, λG is also such a matrix for all $\lambda \in \mathbb{F}^{\times}$. Furthermore G is an arbitrary element of a coset of $\langle D_{\mathbf{w}_i}, T_{\mathbf{w}_i}\rangle \cong 3^{1+12}$, *i.e.*, is defined only up to multiplication by elements of 3^{1+12}.

(5.3.3) Finding Standard Generators

Elements A and B are standard generators of 2·Suz if $A^4 = 1 = B^3$ and $(AB)^{13} = 1$. From standard generators of 2·Suz in $\mathrm{GL}_{12}(3)$ the procedure described in the previous section produces matrices A' and B' in $\mathrm{GL}_{729}(\mathbb{F})$. These are multiples by scalars of elements in the same cosets of 3^{1+12} as are the standard generators. We multiply A' and B' by suitable scalars to produce a matrix A'' of order 4, and a matrix B'' of order 9. Since A^2 commutes with B we have $z = [A''^2, B''] \in 3^{1+12}$ and so

$$[A''^2, B''z] = A''^{-2}(B''z)^{-1}A''^2B''z = (A''^{-2}z^{-1}A''^2)[A''^2, B'']z = z^3 = 1$$

We find that $A''B''z$ has order 13 as required, so we take A'' and $B''z$ as standard generators for the complementary 2·Suz in $\mathrm{GL}_{729}(\mathbb{F})$.

The third generator E is described in [29]. In the terminology of Section 5.1.2 it is an element $(1, \mathbf{v}, \lambda)$ where $\mathbf{v}$ is a vector fixed by a subgroup $\mathrm{U}_5(2)$ of, and negated by the central involution of, 2·Suz. We find such a vector, say $\sum \lambda_i\mathbf{t}_i + \mu_i\mathbf{d}_i$. Hence via the procedure

$$\sum_{i=1}^{6} \lambda_i\mathbf{t}_i + \mu_i\mathbf{d}_i \quad \longmapsto \quad \prod_{i=1}^{6} T_{\mathbf{w}_i}^{\lambda_i} D_{\mathbf{w}_i}^{\mu_i}$$

we produce a matrix $E' \in \mathrm{GL}_{729}(\mathbb{F})$ (`/character/729/do3.sh`), defined up to inversion and multiplication by scalars. As in Section 5.1.2 we use the words from [52] to find the generators U_1 and U_2 for $2 \times \mathrm{U}_5(2)$ as a subgroup of 2·Suz in 729 dimensions over $\mathbb{F}$. Again U_1 and U_2 must both invert E', so we multiply E' by elements of the centre of 3^{1+12} to find the standard generator. We still have the problem that E may need to be replaced with E^{-1}.

(5.3.4) Adjoining The Automorphism: A 1458-dimensional Representation For 3^{1+12}:2·Suz:2

We can extend from 3^{1+12}:2·Suz to 3^{1+12}:2·Suz:2 as follows. The automorphism swaps the standard generators A and B with another pair A' and B', for which words in A and B are given in [52]. We repeat the procedure in the final paragraph of the previous section to find the corresponding third generator E' (`/character/729/do3p.sh`). The automorphism also swaps the 729 dimensional module with its dual, generated by the inverse transpose matrices.

Let $X \in \{A, B, E\}$ and take a direct sum of the representation and its dual to obtain a 1458 dimensional representation with generators $\left(\begin{smallmatrix} X & 0 \\ 0 & X^{-\top} \end{smallmatrix}\right)$. To adjoin the automorphism we require a matrix of the form $\left(\begin{smallmatrix} 0 & C_1 \\ C_2 & 0 \end{smallmatrix}\right)$. Now,

$$\begin{pmatrix} 0 & C_2^{-1} \\ C_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X^{-\top} \end{pmatrix} \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix} = \begin{pmatrix} C_2^{-1} X^{-\top} C_2 & 0 \\ 0 & C_1^{-1} X C_1 \end{pmatrix}$$

$$\text{which we require to be equal to } \begin{pmatrix} X' & 0 \\ 0 & X'^{-\top} \end{pmatrix}$$

and so $C_1^{-1} X C_1 = X'^{-\top}$ and $C_2^{-1} X^{-\top} C_2 = X'$. Inverting and transposing the first of these equations shows that $C_1^{-\top}$ conjugates $X^{-\top}$ to X', *i.e.*, $C_2 = \lambda C_1^{-\top}$ for some scalar $\lambda \neq 0$.

The matrices C_1 and C_2 can be found using the standard basis method and as usual we need to correct the matrices by scalars before pasting them together into a matrix C say. As usual we have an arbitrary element in the correct coset and we need to find standard generators yet again (`/character/729/do5.sh`).

It turns out that B and C^{15} are standard generators for 2·Suz:2, and indeed $C^{30} = A^2$. We now need to find the third standard generator, E' say, in 1458 dimensions. As usual we use the words from [52] to find generators for the subgroup $\mathrm{U}_5(2)$. We first reduce the possibilities for E' from 3^{13} to 3^3 using the 1458-dimensional matrix $F = \left(\begin{smallmatrix} E & 0 \\ 0 & E^{-\top} \end{smallmatrix}\right)$. Under $\mathrm{U}_5(2)$ the 12 dimensional module for 2·Suz:2 splits as

$1+10+1$, so an element $g \in \mathrm{U}_5(2)$ that acts fixed-point-freely on the 10-dimensional module, *e.g.*, an element of class $11A$ of $\mathrm{U}_5(2)$, must fix at most a subgroup 3^{1+2} of 3^{1+12}. To find some elements of this group, consider $(Fg)^{11}$.

$$\begin{aligned}(Fg)^{11} &= Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg \\ &= gF^{g}\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg\,Fg \\ &= g^{11}F^{g^{11}}F^{g^{10}}\dots F^{g}\end{aligned}$$

In the quotient elementary Abelian 3^{12} the F^{g^i} commute, so this is a fixed "vector" of g. Conjugating F by a random element of $\langle C^{15}, B, F\rangle$ and repeating the process we obtain another fixed vector. We find a suitable new standard generator E' by searching in the group generated by these two elements (`/character/729/do7.sh`).

To check our generators we evaluate the words from Section 5.2, inducing the projection map from $3^{1+12}{:}6{\cdot}\mathrm{Suz}{:}2$ to $3^{1+12}{:}2{\cdot}\mathrm{Suz}{:}2$. We found that the word for $(2,1,0)$ produced an element of order 6 instead of 2. This corresponds to the problem described in [29] where C^{15} may have to be replaced with its inverse. Having replaced C^{15} with C^{-15} we know that our standard generators correspond to those in [29]. This allows us to use words in 3 generators to induce the homomorphism to the 196882-dimensional representation constructed in [29] and the 196883-dimensional representation constructed in [50].

(5.3.5) Finding The Character Values

We perform the entire construction described in Section 5.3.1 to Section 5.3.4 over $\mathbb{F}_{103}$ and $\mathbb{F}_{223}$ so that by the Chinese remainder theorem we can compute character values of the degree 1458 character modulo $103.223 = 22969$ (`/character/monster/traces729/traces.mag`). This works because the character values are rational.

We also need the values of the degree 729 characters, but some of these are irrational. We therefore lift eigenvalues to $\mathbb{C}$. To do this we first repeat the construction over $\mathbb{F}_{31}$. For a class representative g we make a field extension that contains all of the eigenvalues, then use the minimum polynomials of the eigenvalues to lift to characteristic zero.

Specifically, we use the Conway polynomial [23, 30] of degree d over $\mathbb{F}_{31}$ to make the field extension, where d is the least integer such that $(31^d - 1)/|g| \in \mathbb{N}$. This extension contains the eigenvalues of g. Let ξ be a primitive element of the field extension, *i.e.,* a root of the Conway polynomial. Then

$$\zeta_d = \xi^{\frac{31^d-1}{|g|}}$$

is the canonical generator of a subgroup of the multiplicative group of the field that contains the eigenvalues of g. Hence we express each of the eigenvalues as a pair (d, n) where ζ_d^n is the eigenvalue.

In characteristic zero the eigenvalue (d, n) lifts to $\exp\left(\frac{2\pi i}{d} n\right)$. The use of Conway polynomials ensures that our eigenvalues are lifted consistently. Thus we avoid having to extend the field so that it contains all eigenvalues of all elements of G: This would be too difficult as such a field would be very large.

(`/character/monster/traces729/tracesEVor.mag`.)

(5.4) Induction From 3^{1+12}:6·Suz

Consider the group $G = 3^{1+12}{:}6{\cdot}\mathrm{Suz}$ for which we have almost found 729-dimensional characters ψ and $\overline{\psi}$ say (we have only evaluated the traces for our class representatives of $G{:}2$). Let $\hat{\eta} \in \mathrm{Irr}\,(6{\cdot}\mathrm{Suz})$ inflate to $\eta \in \mathrm{Irr}\, G$. The Clifford theory in Section 4.1 tells us that $\psi\eta{\uparrow}G{:}2$ is an irreducible character. The outer automorphism fuses ψ with $\overline{\psi}$ to give

$$\chi(g) = \begin{cases} \psi(g) + \overline{\psi}(g) & \text{if } g \text{ is in the inner half of } G{:}2 \\ 0 & \text{if } g \text{ is in the outer half of } G{:}2 \end{cases}$$

Moreover, η may either split or fuse:

- In the splitting case η extends in two ways, giving characters η' and $\overline{\eta}'$ say, which are equal to η on inner elements. We thus obtain

$$\begin{aligned} \langle \chi\eta', \chi\eta' \rangle_{G:2} &= \frac{1}{2|G|} \sum_{g \in G} \chi(g)\eta'(g)\overline{\chi\eta'(g)} + \frac{1}{2|G|} \sum_{g \in G:2 \setminus G} \chi(g)\eta'(g)\overline{\chi(g)\eta'(g)} \\ &= \frac{1}{2|G|} \sum_{g \in G} (\psi(g) + \overline{\psi}(g))\eta(g)\overline{(\psi(g) + \overline{\psi}(g))\eta(g)} + \frac{1}{2|G|} \sum_{g \in G:2 \setminus G} 0\eta'(g)\overline{0\eta'(g)} \\ &= \tfrac{1}{2} \left\langle (\psi + \overline{\psi})\eta, (\psi + \overline{\psi})\eta \right\rangle_G \\ &= 1 \end{aligned}$$

so the character $\chi\eta'$ is irreducible. As χ is zero on outer elements we have $\chi\eta' = \chi\overline{\eta}'$. We obtain 47 more irreducible characters of G:2 in this way.

- In the fusion case we obtain a single new character η' which takes values $\eta(g) + \overline{\eta}(g)$ if g is inner and $\eta'(g) = 0$ if g is outer. Hence

$$\begin{aligned}\langle \chi\eta', \chi\eta' \rangle_{G:2} &= \frac{1}{2|G|} \sum_{g \in G} (\psi(g) + \overline{\psi}(g))(\eta(g) + \overline{\eta}(g))\overline{(\psi(g) + \overline{\psi}(g))(\eta(g) + \overline{\eta}(g))} + 0 \\ &= \tfrac{1}{2} \left\langle (\psi + \overline{\psi})(\eta + \overline{\eta}), (\psi + \overline{\psi})(\eta + \overline{\eta}) \right\rangle_G \\ &= 2\end{aligned}$$

so $\chi\eta'$ is not irreducible. However, we can find the two irreducible constituents: Corollary 6 on page 33 tells us they are $\psi\eta\uparrow G{:}2(g) = \psi\eta(g) + \overline{\psi\eta}(g)$ and $\overline{\psi}\eta\uparrow G{:}2(g) = \overline{\psi}\eta(g) + \psi\overline{\eta}(g)$ (g inner, both are zero on outer elements). The values of these characters can be evaluated as we have identified the conjugacy classes of 6·Suz and the fusion map to 6·Suz:2.

We obtain a further 163 irreducible characters in this way.

(5.5) Induction From The Index 65520 Inertia Group

We shall induce characters from the subgroup 3^{1+12}:((3 × U$_5$(2)):2) as it has smaller index than 3^{1+12}:(3.3^5:(M$_{11}$ × 2)):2. We shall determine the eigenvalues of our representing matrices and lift them to $\mathbb{C}$ as described in Section 5.3.5.

We are not interested in representing the centre of the extraspecial group faithfully, so for the rest of this section we shall work with $G = 3^{12}$:6·Suz:2. Our inertia group becomes 3^{12}:((3 × U$_5$(2)):2) which is explained in section 5.5.2. We then describe the representations we shall construct and how we induce these to G.

(5.5.1) A Permutation Representation of 3^{12}:6·Suz:2

In Section 5.1.2 we made a 14-dimensional representation of 3^{1+12}:2·Suz:2 with generators C_{14}, D_{14}, and E_{14} that are matrices of the form shown in equation (17). The corresponding module consists of the vectors of $\mathbb{F}_3^{14}$ in which the last 13 co-ordinates form a submodule on which the quotient group 3^{12}:2·Suz:2 is represented. Deleting the first row and column of our three 14-dimensional matrices gives us matrices, C_{13}, D_{13}, and E_{13} say, representing the action of 3^{12}:2·Suz:2 on this 13-dimensional module.

Let $\mathbf{e}$ be the fixed vector of $U_5(2)$ found in Section 5.1.2. The space $\langle \mathbf{e} \rangle$ has 32760 images under the action of 2·Suz:2 on the quotient 12-dimensional module. In the 13-dimensional module there are $3 \times 32760 = 98280$ images of $\langle (\mathbf{e}\, 0) \rangle$. To see this recall the matrix S^{-1} from Section 5.1.2 that represents the invariant symplectic form, and let $\mathbf{v} \in \mathbb{F}_3^{12}$ have $\mathbf{v}S^{-1}\mathbf{e}^\top \neq 0$. The 13-dimensional matrix

$$\begin{pmatrix} I_{12} & -S^{-1}\mathbf{v}^\top \\ \mathbf{v} & 1 \end{pmatrix} \in 3^{12}{:}2{\cdot}\mathrm{Suz}{:}2$$

thus has orbit $\{(\mathbf{e}, 0), (\mathbf{e}, 1), (\mathbf{e}, -1)\}$ on the 13-dimensional module. Therefore the stabiliser in 3^{12} of $\langle (\mathbf{e}\, 0) \rangle$ is precisely those matrices of the form given above but for which $\mathbf{v} \in \langle (\mathbf{e}\, 0) \rangle^\perp$ (with respect to S^{-1}) and so the full stabiliser is 3^{11}:(2 × $U_5(2)$:2).

Using the MeatAxe vector permute program (`zvp -p`) we obtain permutations $\hat{C}, \hat{D}, \hat{E}$ for the actions of C_{13}, D_{13}, E_{13} on 98280 points. This is a representation of 3^{12}:2·Suz:2.

There is a representation of 6·Suz:2 on 5346 points available on the web Atlas on standard generators $\tilde{C}$ and $\tilde{D}$ say. We re-number the 5346 points as 98281 to 103626 and make permutations $C = \hat{C}\tilde{C}$, $D = \hat{D}\tilde{D}$, and $E = \hat{D}\mathrm{Id}_{\langle \tilde{C}, \tilde{D} \rangle}$ on 103626 points representing $G = 3^{12}$:6·Suz:2. The stabiliser of point 1 becomes 3^{11}·(2 × (3 × $U_5(2)$):2) which is extended to $H = 3^{12}$:(2 × (3 × $U_5(2)$):2) by E^D. The inertia group has index 2 in H.

We can obtain the 32760 point representation, π say, of 3^{12}:6·Suz:2 on the cosets of H as follows (π represents faithfully the quotient Suz:2). The orbits of E^D are blocks of imprimitivity in the 98280 point representation and fix all points in the 5346 point representation. An element of 2·Suz:2 acts as ± 1 on the 13th co-ordinate so the non-trivial blocks are preserved by C and D too. The resulting representation π on 32760 points is the action on the blocks and the action on the cosets of H.

(5.5.2) The Group 3^{12}:(2 × (3 × $U_5(2)$):2) And A Quotient Of It

The inertia factor group is of course (3 × $U_5(2)$):2 and we would like to lift a representation of this to the full inertia group, tensor it with an irreducible representation of 3^{12} extended to and fixed by the inertia group, and then induce to G.

But induction is transitive and the inertia group is a subgroup of index 2 of H for which we have the representation on cosets, π. We may write

$$H = 3^{11}{}^{\cdot}(\mathrm{S}_3 \times (3 \times \mathrm{U}_5(2)){:}2)) \qquad \text{and} \qquad \overline{H} = \mathrm{S}_3 \times (3 \times \mathrm{U}_5(2)){:}2)$$

The quotient $\overline{H}$ represents E^D and not E faithfully, but this will not matter as the induction sums over all such conjugates. Recall that the 12-dimensional $\mathbb{F}_3$-module for $6{}^{\cdot}\mathrm{Suz}{:}2$ splits up as $1+10+1$ when restricted to $2 \times (3 \times \mathrm{U}_5(2)){:}2$. The submodule 3^{11} is that exhibited in the shape of H above. The standard generator E is in the 1-dimensional submodule, and it turns out that E^D extends 3^{11} to 3^{12}. We write

$$\overline{H} = \langle e', c_2, c_3, u_1, u_2 \rangle \tag{24}$$

where u_1 and u_2 are standard generators for $\mathrm{U}_5(2){:}2$, e' is the image of E^D, and c_2 and c_3 are elements in the centre of $6{}^{\cdot}\mathrm{Suz}$ of orders 2 and 3 respectively. We shall induce an irreducible representation of $\overline{H}$ lifted to H up to G, provided that e' is represented faithfully.

We can easily find H inside our 103626 point permutation representation for G. $E^D \in G$ represents e'. We find permutations C_2 and C_3 representing c_2 and c_3 by using the words for these from Section 5.1. To find the permutations U_1 and U_2 representing u_1 and u_2 we use the words from the web Atlas which give us generators for $2 \times \mathrm{U}_5(2){:}2$. We can tell that this gives us C_2U_1 instead of U_1 because C_2U_1 conjugates E to its inverse whereas U_1 centralises it.

Having found these generators for H we can induce the natural homomorphism to $\overline{H}$ by evaluating words in them.

(5.5.3) Representations of $\mathrm{S}_3 \times (3 \times \mathrm{U}_5(2)){:}2)$

The exponent of $\overline{H}$ is equal to the exponent of $\mathrm{U}_5(2){:}2$ which is 7920. It is very fortunate that $7920 = 89^2 - 1$, so it will be sufficient to work over $\mathbb{F}_{89^2}$ which will contain all the eigenvalues.

Observe that

$$\mathrm{S}_3 = \langle s_3, s_2 | s_2^2 = 1 = s_3^3,\ s_3^{s_2} = s_3^{-1} \rangle \cong \langle \overline{S}_3, \overline{S}_2 \rangle \text{ where } \overline{S}_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } \overline{S}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

over any field of characteristic not 2.

Suppose we have constructed standard generators $\overline{U}_1$ and $\overline{U}_2$ for $\mathrm{U}_5(2){:}2$ (we shall do this in the next section) in n dimensions over $\mathbb{F}_{89^2}$. The following table describes three modules M_1, M_2, and M_3.

	e'	c_2	c_3	u_1	u_2
M_1	I_n	I_n	I_n	$\overline{U}_1$	$\overline{U}_2$
M_2	I_2	I_2	$\overline{S}_3$	$\overline{S}_2$	$\overline{S}_2$
M_3	$\overline{S}_2$	$\overline{S}_2$	I_2	I_2	I_2

The columns are labeled with the generators of $\overline{H}$ as named in equation (24). The rows are labeled with the module names. Each entry of the body of the table is the representation of a generator on a module. Recall that we must represent e' faithfully, so we are interested in the modules $M = M_1 \otimes M_2 \otimes M_3$ and $M_1 \otimes M_3$.

Note that the inclusion of c_2 in H fuses the irreducible modules of 3^{12} in pairs. This is why we have a 2-dimensional module, M_3, for $\langle e', c_3 \rangle$ instead of a 1-dimensional module for $\langle e' \rangle$.

However, M is not necessarily irreducible. If M_1 splits into two irreducibles when restricted to $\mathrm{U}_5(2)$ then write $M_1 = V + \overline{V}$ as modules for $\mathrm{U}_5(2)$, and write $M_2 = W + \overline{W}$ where W and $\overline{W}$ are irreducible modules for $\langle \overline{S}_3 \rangle$. As modules for $3 \times \mathrm{U}_5(2)$ we have

$$(V + \overline{V}) \otimes (W + \overline{W}) = (V \otimes W + \overline{V} \otimes \overline{W}) + (V \otimes \overline{W} + \overline{V} \otimes W)$$

where the brackets on the right hand side indicate the fusion in $(3 \times \mathrm{U}_5(2)){:}2$, in particular $M_1 \otimes M_2$ is reducible. We must tensor only one of these irreducibles with M_3 before induction, and we can use Magma's MeatAxe to find the irreducibles. Note that the module $M = M_1 \otimes M_3$ *is* irreducible.

Alternatively, $M_1 {\downarrow} \mathrm{U}_5(2)$ is irreducible but extends to $\mathrm{U}_5(2){:}2$ in two ways. We obtain the action on other extension module, M_1^- say, by multiplying $\overline{U}_1$ and $\overline{U}_2$ by -1 (both lie in the outer half of the group). However, we can find the character induced from $M_1^- \otimes M_3$ by tensoring the character induced from $M_1 \otimes M_3$ with the linear character that takes value -1 on outer elements. The characters induced from $M_1^- \otimes M_2 \otimes M_3$ and $M_1 \otimes M_2 \otimes M_3$ are identical—the inclusion of M_2 gives them value 0 on outer elements. We shall make use of this later to speed up the induction calculations.

We now have to find some representations of $\mathrm{U}_5(2){:}2$ over $\mathbb{F}_{89^2}$.

(5.5.4) Some Representations Of $U_5(2){:}2$

We obtain some modules with almost no effort by converting the permutation representations available on the web Atlas into permutation matrices over $\mathbb{F}_{89}$ and chopping with the MeatAxe. We were fortunate to obtain a 10-dimensional representation over $\mathbb{Z}[\sqrt{-2}]$ [6]. We find that $\sqrt{-2}$ reduces to $40 \in \mathbb{F}_{89} \leqslant \mathbb{F}_{89^2}$. The 22-dimensional representation over $\mathbb{Z}$ is available on the web Atlas, and is easy to reduce to $\mathbb{F}_{89^2}$.

With this good selection of modules already available it is easy to find the remaining modules by tensoring the modules we have and chopping. We use the character table of $U_5(2){:}2$ in GAP to tensor and reduce characters thus guiding our search.

A small selection of these modules ready for use in our induction program, and the programs we used to make them, are on the CD at `/character/induction/modules/`.

(5.5.5) The Induction

Let T be a transversal for H in G. If our modules M_i afford characters χ_i and M affords character χ then $\chi(g) = \chi_1(g)\chi_2(g)\chi_3(g)$ and $\chi\!\uparrow\!G(g) = \sum_{t\in T} \dot{\chi}(g^t)$. The permutation representation π of G on the cosets of H was calculated in Section 5.5.1. Each $t_i \in T$ corresponds to a coset t_iH which in turn corresponds to a point i. Now, $t_igt_i^{-1} \in H$ if and only if $t_igt_i^{-1}$ fixes 1 which means that g fixes i. Hence

$$\chi\!\uparrow\!G(g) = \sum_{i\in \mathrm{Fix}\,\pi(g)} \chi(t_igt_i^{-1})$$

The transversal element t_i is an element of G for which $1\pi(t_i) = i$. We do not store all 32760 elements of T, or even 32760 words for them. Instead we make a Schreier vector $\mathbf{v}$ and backward pointer vector $\mathbf{w}$ as described in Chapter 7 of [7]. We have a list $\mathbf{g}$ of elements of G. To construct $\mathbf{v}$ and $\mathbf{w}$ we begin at point 1 and enumerate the orbit $1^{\pi(G)}$ breadth first using the generators in $\mathbf{g}$. The backward pointer vector $\mathbf{w}$ holds the structure of a tree for which $\mathbf{v}$ holds labels for the edges. Vertex $i\pi(\mathbf{g}_j)$ is a child of vertex i so $\mathbf{w}_{i\pi(\mathbf{g}_j)} = i$. Therefore vertex $\mathbf{w}_i$ is the parent to vertex $\mathbf{w}_{i\pi(\mathbf{g}_j)}$, hence the term "backward pointer". In the Schreier vector $\mathbf{v}_{i\pi(\mathbf{g}_j)} = j$ so that we know which element of $\mathbf{g}$ takes us down the tree from vertex i to vertex $i\pi(\mathbf{g}_j)$. To find an element $\pi(t_i)$ that maps 1 to i we begin at vertex i and find that $i = \mathbf{w}_i\pi(\mathbf{g}_{\mathbf{v}_i})$, and continue in this way to the top of the tree. The transversal element t_i can be obtained at the same time by taking a product of the $\mathbf{g}_{\mathbf{v}_i}$ instead of $\pi(\mathbf{g}_{\mathbf{v}_i})$.

We can now proceed with the induction as follows:

1. Obtain a representative $g \in G$ for one of the 533 conjugacy classes of $3^{1+12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$. We found words for the conjugacy class representatives in Section 5.1.3.

2. For each $x \in \mathrm{Fix}\,\pi(g)$ use the Schreier vector $\mathbf{v}$ and backward pointers $\mathbf{w}$ to produce a transversal element $t_i \in G$.

3. Find a word in $\langle E, C_2, C_3, U_1, U_2\rangle$ for $t_i g t_i^{-1}$ and evaluate it in $\langle \overline{E}, \overline{C}_2, \overline{C}_3, \overline{U}_1, \overline{U}_2\rangle$ to produce a matrix h with entries in $\mathbb{F}_{89^2}$. The evaluation of these words takes most of the running time of the program. In some cases we could reduce the evaluation time to about $\frac{2}{3}$ by putting the generators of M_1 into a standard basis, thus increasing their sparsity.

4. Find the eigenvalues of h and write them as powers of ζ where ζ is a primitive element of $\mathbb{F}_{89^2}$.

If g lies above the centre of $6{}^{\cdot}\mathrm{Suz}$ then $\pi(g)$ fixes all 32760 points and the induction takes a very long time. A closer examination of these cases reveals that in fact we can just write down the traces. There are 14 such cases:

(i) The classes labelled $(1,1,0)$ and $(1,1,1)$ which lie above the identity of $3^{12}{:}6{}^{\cdot}\mathrm{Suz}{:}2$.

(ii) The classes labelled $(1,2,0)$ and $(1,3,0)$ which lie above above the identity of $6{}^{\cdot}\mathrm{Suz}{:}2$ but are not conjugate to any element of the complementary $6{}^{\cdot}\mathrm{Suz}{:}2$. (These classes correspond to the orbits of length 32760 and 465920 of $6{}^{\cdot}\mathrm{Suz}{:}2$ on 3^{12}).

(iii) Classes labelled $(2,1,0)$ and $(2,1,1)$ lying above the central involution of $6{}^{\cdot}\mathrm{Suz}{:}2$, and classes labelled $(21,1,0)$, $(21,1,1)$, and $(21,1,2)$ which map onto the central element of order 6 in $6{}^{\cdot}\mathrm{Suz}$.

(iv) The classes labelled $(6,1,0)$, $(6,1,1)$ and $(6,1,2)$ lying above the central 3-elements of $6{}^{\cdot}\mathrm{Suz}$ and having conjugates in the complement.

(v) The classes labelled $(6,2,0)$ and $(6,3,0)$ lying above the central 3-elements of $6{}^{\cdot}\mathrm{Suz}$ and with no conjugates in the complement.

In case (iii) the central involution is represented as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on M_3 so the trace on both M and $M_1 \otimes M_3$ is zero.

On $M_1 \otimes M_3$ in cases (i) and (iv) the trace is clearly $32760 \times \dim M_1 \times \dim M_3$. For M irreducible we include a factor of 2 for case (i) and a factor of -1 for case (iv). If M splits then we also include these factors, but instead multiply these with $\frac{1}{2} \times 32760 \times \dim M_1 \times \dim M_3$.

In cases (ii) and (v) the elements have order 3 and so the character value of one is $a - b$ where a is the multiplicity of the eigenvalue 1 and $\frac{1}{2}b$ is the multiplicity of the eigenvalue $\exp\left(\frac{2\pi i}{3}\right)$. In each case we calculate a and b for M_1 the trivial module. But M_1 always represents such elements trivially, so the character value for arbitrary M_1 is simply $(a - b) \times \dim M_1$. The values of a and b are different between the three cases $M_1 \otimes M_3$, M irreducible, and M splits into two irreducibles. We perform the full induction for one of each of these cases and so deduce the values of a and b.

The induction programs and supporting files are on the CD at `/character/induction/`. Their use is described in Appendix A.4.

(5.6) Conclusions

We have the following further sources of characters:

- Characters restricted from $\mathbb{M}$ to $N(3B)$ then lifted to 3^{1+12}:6·Suz:2.
- Characters induced from the subgroup 6·Suz:2.
- Symmetric squares, anti-symmetric squares, and tensor products of pairs of our known irreducibles.

We reduce these characters using our known irreducibles, and sometimes this yields a new irreducible. We have now found 463 irreducible characters.

We are continuing the induction calculations in order to find more new irreducibles and hopefully complete the character table. Having found all of the irreducible characters of 3^{1+12}:6·Suz:2 we would use our explicitly computed quotient map to project characters with the correct kernel down to $N(3B)$ and so obtain the character table we desire. However, we also know the projection maps to 3^{12}:6·Suz:2 and 3^{1+12}:2·Suz:2 and 3^{12}:2·Suz:2, so we could also produce the character tables of these groups.

The following conjugacy classes of $\mathbb{M}$ intersect $N(3B)$ non-trivially:

1A	2A	2B	**3A**	3B	**3C**	4A	4B	4C	4D	5A	5B
6A	6B	**6C**	6D	6E	**6F**	7B	8A	**8B**	8C	8D	8E
8F	9A	9B	10B	10C	10D	10E	11A	**12A**	12B	**12C**	**12D**
12E	12F	12G	12H	12I	**12J**	13B	14C	**15A**	15B	15C	**15D**
16A	18A	18B	18C	18D	18E	20C	20D	**20E**	20F	**21B**	21D
22B	**24A**	**24B**	24C	**24D**	**24E**	24F	24G	24H	24I	**24J**	26B
27A	27B	28D	30A	**30C**	30D	**30E**	30F	30G	33A	**33B**	36A
36B	36C	36D	39C	39D	**40C**	**40D**	42B	**42D**	45A	**48A**	54A
56B	**56C**	60C	60D	60E	66B	78B	78C	84B			

The classes marked in bold were not known in Section 3.5.

Chapter 6

Nets

Nets were invented by Norton in 1987 [37] as a way to encode information about certain subgroups of the Monster in a way sympathetic with the connection to modular functions. Norton initiated a classification of nets but there are far too many nets to classify by hand (there are 1400384 conjugacy classes of flags), and a complete classification was dismissed as probably infeasible [39]. Nets are connected with a non-commutative version of Norton's "generalised moonshine" [37] which is thought to be the fullest generalisation of the Moonshine conjectures. In this chapter we work towards a complete enumeration of nets.

Computer programs for this chapter can be found on the CD at `/nets`. The contents of this folder are described in Appendix A.5.

(6.1) Definitions And Terminology

Our notation and terminology follows [39]: Let $\tilde{\Gamma}$ be the three string braid group, so

$$\tilde{\Gamma} = \langle x, y \mid xyx = yxy \rangle = \langle s, t \mid s^3 = t^2 \rangle$$

where $s = xy$ and $t = xyx$. The element $z = s^3 = t^2$ is central in $\tilde{\Gamma}$ and $\tilde{\Gamma}/\langle z^2 \rangle \cong \mathrm{SL}_2(\mathbb{Z})$. Quotienting out $\langle z \rangle$ leaves the modular group, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$.

Let (a, b, c) be a triple of involutions in $\mathbb{M}$, each in class $2A$. An action of $\tilde{\Gamma}$ on the set of such triples is obtained by interpreting the braiding action "passing a under b" as $(a, b, c) \mapsto (b, a^b, c)$, and "passing b under c as $(a, b, c) \mapsto (a, c, b^c)$. Hence

$$x\colon (a,b,c) \mapsto (b, a^b, c) \qquad s\colon (a,b,c) \mapsto (b, c, a^{bc}) \qquad t\colon (a,b,c) \mapsto (c, b^c, a^{bc}) \qquad z\colon (a,b,c) \mapsto (a,b,c)^{abc}$$

(maps are written on the right) where $(a,b,c)^g$ is shorthand for (a^g, b^g, c^g).

A polyhedron can be associated with this action as follows. Let a vertex be an orbit of s, an edge be a t-orbit, and a face be an $\langle x, z\rangle$-orbit. The $\langle z\rangle$-orbits are then the flags of the resulting polyhedron. Thus the polyhedron is defined modulo the action of $\langle z\rangle$, *i.e.*, we may work with Γ rather than $\tilde{\Gamma}$.

Definition 25 *For a triple* (a,b,c) *of* $2A$ *involutions of* $\mathbb{M}$ *the net* $\mathcal{N}(a,b,c)$ *is the geometry obtained by applying the operations s and t to* (a,b,c).

A simple example is a net generated from involutions

$$a = (1\,2)(3\,4) \qquad b = (1\,4)(2\,3) \qquad c = (1\,2)(4\,5)$$

which generate A_5. The net is displayed in Figure 5. Each flag contains five triples and we label the flags as follows, giving just a representative triple for flags 5 to 10:

$$\begin{aligned} f_1 = \{&((1\,2)(3\,5),(1\,3)(2\,5),(1\,4)(3\,5)),\\ &((2\,4)(3\,5),(2\,3)(4\,5),(1\,3)(2\,4)),\\ &((1\,5)(2\,4),(1\,4)(2\,5),(1\,5)(2\,3)),\\ &((1\,5)(3\,4),(1\,3)(4\,5),(2\,5)(3\,4)),\\ &((1\,2)(3\,4),(1\,4)(2\,3),(1\,2)(4\,5))\} \end{aligned} \qquad \begin{aligned} f_2 = \{&((1\,3)(2\,5),(1\,4)(3\,5),(2\,4)(3\,5)),\\ &((2\,3)(4\,5),(1\,3)(2\,4),(1\,5)(2\,4)),\\ &((1\,4)(2\,5),(1\,5)(2\,3),(1\,5)(3\,4)),\\ &((1\,3)(4\,5),(2\,5)(3\,4),(1\,2)(3\,4)),\\ &((1\,4)(2\,3),(1\,2)(4\,5),(1\,2)(3\,5))\} \end{aligned}$$

$$\begin{aligned} f_3 = \{&((1\,4)(3\,5),(2\,4)(3\,5),(2\,3)(4\,5)),\\ &((1\,3)(2\,4),(1\,5)(2\,4),(1\,4)(2\,5)),\\ &((1\,5)(2\,3),(1\,5)(3\,4),(1\,3)(4\,5)),\\ &((2\,5)(3\,4),(1\,2)(3\,4),(1\,4)(2\,3)),\\ &((1\,2)(4\,5),(1\,2)(3\,5),(1\,3)(2\,5))\} \end{aligned} \qquad \begin{aligned} f_4 = \{&((1\,4)(3\,5),(2\,3)(4\,5),(2\,4)(3\,5)),\\ &((1\,3)(2\,4),(1\,4)(2\,5),(1\,5)(2\,4)),\\ &((1\,5)(2\,3),(1\,3)(4\,5),(1\,5)(3\,4)),\\ &((2\,5)(3\,4),(1\,4)(2\,3),(1\,2)(3\,4)),\\ &((1\,2)(4\,5),(1\,3)(2\,5),(1\,2)(3\,5))\} \end{aligned}$$

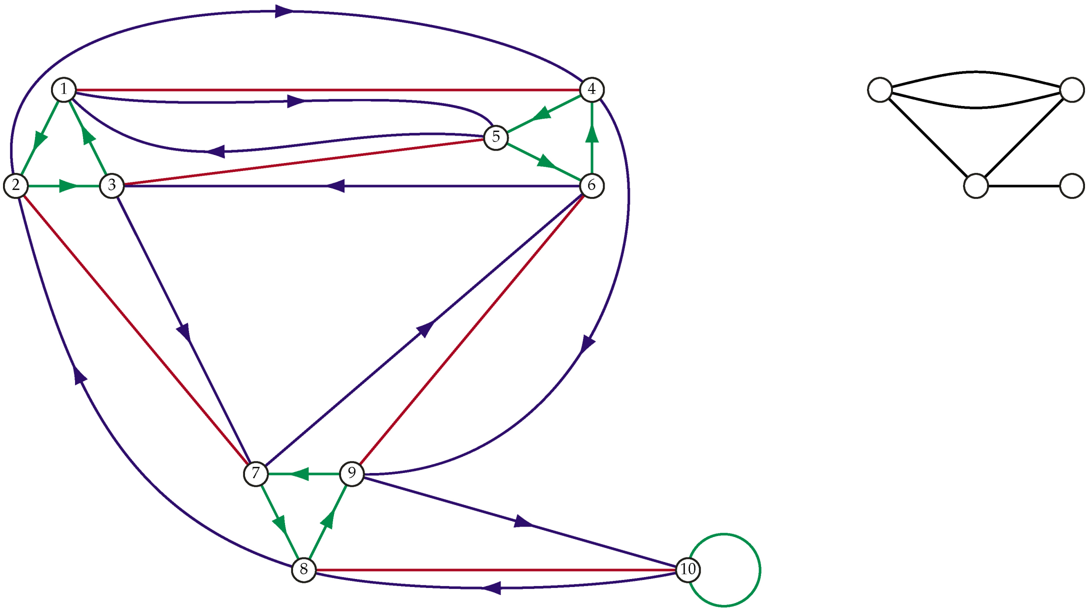


Figure 5: A net. On the left is an "exploded view" showing the flags as labelled in the equations on page 78. The actions of s and x are shown in green and blue. The action of t is shown by the red edges, but one un-directed edge is displayed instead of two directed edges (one in each direction). On the right is the actual net. Flag 10 is fixed by s and so becomes a collapsed vertex.

$$f_5 \ni ((2\,3)(4\,5), (2\,4)(3\,5), (1\,3)(2\,4)) \qquad f_6 \ni ((2\,4)(3\,5), (1\,3)(2\,4), (1\,4)(2\,5))$$

$$f_7 \ni ((2\,4)(3\,5), (1\,2)(3\,5), (2\,3)(4\,5)) \qquad f_8 \ni ((1\,2)(3\,5), (2\,3)(4\,5), (1\,5)(2\,4))$$

$$f_9 \ni ((2\,3)(4\,5), (1\,5)(2\,4), (2\,4)(3\,5)) \qquad f_{10} \ni ((1\,5)(2\,4), (1\,2)(3\,4), (2\,4)(3\,5))$$

We can now list the vertices, edges, and faces as orbits on flags. This induces a homomorphism $\Gamma \to S_{10}$

$$s \mapsto (1\,2\,3)(4\,5\,6)(7\,8\,9)(10)$$

$$t \mapsto (1\,4)(2\,7)(3\,5)(6\,9)(8\,10)$$

$$x \mapsto (1\,5)(2\,4\,9\,10\,8)(3\,7\,6)$$

where the permutations act on the indices i of our flags f_i.

We store a net $\mathcal{N}(a, b, c)$ as:

(i) A list F of flags.

(ii) A list of vertices, each being a list of flags.

(iii) A list of edges. This may be stored as a list of flags, or a list of vertices.

(iv) A list of faces. This may also be stored as a list of flags, or a list of vertices.

In Section 6.3.2 we shall make this more explicit, defining some Magma record formats and describing some programs. Of course if (a', b', c') is a triple contained in any element of F and if F' is the set of flags of $\mathcal{N}(a', b', c')$ then $F = F'$ (as sets) and so we say $\mathcal{N}(a, b, c) = \mathcal{N}(a', b', c')$. Moreover, if $g \in \mathbb{M}$ then $\mathcal{N}(a^g, b^g, c^g)$ is conjugate to $\mathcal{N}(a, b, c)$ in the sense that g conjugates F to the set of flags of $\mathcal{N}(a^g, b^g, c^g)$ and preserves the net structure. We wish to find representatives for every $\mathbb{M}$-conjugacy class of nets.

Definition 26 *We use the following terms applied to the net $\mathcal{N}(a, b, c)$.*

(i) The class *of $\mathcal{N}(a, b, c)$ is the conjugacy class in $\mathbb{M}$ of abc. This is an invariant of the net since $abc \overset{x}{\mapsto} ba^bc = abc$ and $abc \overset{y}{\mapsto} acb^c = abc$.*

(ii) The order *of $\mathcal{N}(a, b, c)$ is the order of abc.*

(iii) The order *of the face containing the flag containing the triple (a, b, c) is the order of ab. This is well defined because $ab \overset{x}{\mapsto} ba^b = ab$.*

(iv) The modulus *of the net is the smallest $n \in \mathbb{N}$ such that $(abc)^n$ commutes with each of a, b, and c.*

In Figure 5 we see that the order of the net is $|abc| = 5$, and since each flag contains five triples the modulus is 5 too. The net group is A_5 and from Norton's list [39] we see that this net could be either $5a, 1$ or $5a, 2$ depending on whether we embed our A_5 in the $11A$ centraliser or the $19A$ centraliser (respectively). In the former case this net will turn up later inside M_{24} and receive the label `11:34:[71]`. In the latter case it will turn up in S_5 and receive label `19:6:[8]`.

Definition 27 *Since $[\langle s\rangle : \langle z\rangle] = 3$ a vertex usually belongs to 3 flags (and so has degree 3). However, sometimes the vertex will* collapse *and belong to only 1 flag. Similarly,*

- *an edge collapses if it belongs to 1 rather than 2 flags. A collapsed edge should not be confused with a loop: A loop has two ends in the same vertex whereas a collapsed edge only has one end!*
- *a face collapses if the number of flags it belongs to (its* collapsed order*) is less than its order.*

We see a collapsed vertex—containing only flag 10—in Figure 5.

Definition 28 *The* genus *of a net is the genus of the lowest genus surface on which it may be drawn.*

Since $2A \subseteq \mathbb{M}$ is a class of 6-transpositions, all faces have at most 6 sides and so the genus is either 0 or 1. Clearly the net in Figure 5 has genus 0.

To assist in calculating the genus of a net, each vertex, edge, and face may be given a "defect" in a way compatible with Euler's formula so that the sum of the defects is the genus of the net. Write p_i for the number of vertices with i edges and r_i for the number of faces with i sides. If q is the number of edges then the Handshaking Lemma tells us that

$$\sum_i ip_i = 2q = \sum_i ir_i$$

Now let p be the number of edges and R be the set of faces and recall Euler's formula $p - q + |R| = 2(1-g)$ for genus g. From the Handshaking Lemma we obtain

$$12(1-g) = 2\sum_i (3-i)p_i + \sum_i (6-i)r_i$$

It is clear from this that an un-collapsed vertex should have defect 0 and a face should have defect $6-i$ where i is its collapsed order. Collapsing a vertex reduces p_3 by 1 and increases p_1 by 1, so a collapsed vertex should have defect $2(-0+2) = 4$. If an edge collapses then the hypotheses used in proving the Handshaking Lemma do not hold. The problem with a collapsed edge is that it has just one end (not two) and one side. Hence when an edge collapses we must have

$$\sum_i ip_i = 2q - 1 = \sum_i ir_i$$

and so we obtain

$$12(1-g) = 2\sum_i (3-i)p_i + \sum_i (6-i)r_i - 3$$

and so the defect of a collapsed edge should be 3. The defect of an un-collapsed edge is 0. Now let the net have n flags, v collapsed vertices, e collapsed edges, and let $r \in R$ have $|r|$ sides, so

$$\begin{aligned} 2(1-g) &= \frac{2v}{3} + \frac{e}{2} + \sum_{r\in R}\left(1 - \frac{|r|}{6}\right) \\ &= \frac{2v}{3} + \frac{e}{2} + |R| - \sum_{r\in R}\frac{|r|}{6} \\ &= \frac{2v}{3} + \frac{e}{2} + |R| - \frac{n}{6} \end{aligned} \tag{29}$$

This is Theorem 6.2.1 of [20], obtained rather more easily.

For consistency we always embed nets in space in the following way: Applications of the operation s will correspond to going round the flags of a vertex anti-clockwise. Similarly, x shall go round a face anti-clockwise.

Definition 30 *The* exponent *of a vertex (resp. edge, face) is defined as the power of abc by which one conjugates when going all the way round clockwise,* i.e., *when applying to (a, b, c) the operation s^{-1} (resp. t, x^{-1}) until one first returns to the original flag. The exponent of a net is the sum of the exponents of all its vertices, edges, and faces.*

We have made a number of choices about how to draw nets: s^{-1} takes one round a vertex clockwise for example. Accordingly, the net rotation group is the group of orientation preserving automorphisms which can be realised as conjugation by an element of $\mathbb{M}$. There may be more such automorphisms that reverse the orientation (and in doing so invert abc). Adjoining these gives the net reflection group.

(6.2) Moonshine

(6.2.1) The Original Moonshine Conjectures

The following, and of course much more, is described in the original moonshine paper, [9].

The modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ acts on the upper half complex plane, $\mathfrak{H}$, as Möbius transformations, *i.e.*,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

and these transformations leave invariant the function field $\mathbb{C}(j)$ where j is the elliptic modular function [3] which has Fourier expansion

$$j = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \qquad = \sum_{i=-1}^{\infty} a_i q^i \qquad (q = e^{2\pi i z})$$

Usually the normalised function $J = j - 744$ is preferred. J. McKay noticed that $196884 = 196883 + 1$, the summands on the right being the degrees of the two smallest representations of the Monster. The other coefficients are also non-negative integer linear combinations of Monster character degrees and so one can replace the coefficients with the characters $H_i(g)$ (or representations) they suggest. This gives a function

$$T_g = q^{-1} + 0 + H_1(g)q + H_2(g)q^2 + \dots \tag{31}$$

for each conjugacy class of the Monster, with $J = T_1$. It was conjectured that T_g is the normalised generator of a genus zero function field arising from a subgroup between $\Gamma_0(n)$ and its normaliser in $\mathrm{PSL}_2(\mathbb{R})$. (Here $\Gamma_0(n)$ is the subgroup consisting of elements with c divisible by n.)

(6.2.2) Generalised Moonshine And The Link With Nets

Let g and h be commuting elements of the Monster. In [37] Norton conjectures that associated with each such pair is a meromorphic function F with the following properties:

1. $F(g^a h^c, g^b h^d, z) = F\left(g, h, \dfrac{az+b}{cz+d}\right)$ when $ad - bc = 1$.
2. Fix g and expand F as powers of $q = e^{2\pi i z}$,

   $$F(g, h, z) = \sum_{i=1}^{\infty} H_i(h) q^i$$

   then for each i the coefficient function $H_i(h)$ is, up to multiplication by a root of unity, a projective character of $C_{\mathbb{M}}(g)$.
3. Simultaneous conjugation of g and h by an element of the Monster leaves F invariant.
4. Unless F is constant its invariance group is a modular group of genus 0.

This is the generalised moonshine conjecture, which links pairs of commuting elements in the Monster with modular functions that are a "Hauptmodul" (generator) for a genus zero function field. In particular, choosing $g = 1$ gives the original Moonshine functions. These conjectures have been proven by C. Dong, H. Li, and G. Mason in the case where $\langle g, h\rangle$ is cyclic [10].

A natural further generalisation is the case when g and h do not commute. Norton has indicated[1] that in general non-commutative moonshine does not hold. His exception to this is the special case where $a, b, c \in \mathbb{M}$ are $2A$-involutions and $g = ab$ and $h = bc$. This is thought to be the fullest generalisation of the moonshine conjectures.

For each net we obtain a subgroup $\Delta \leqslant \Gamma$ as the stabiliser of a flag. By parts 1 and 3 of the Generalised Moonshine Conjectures it is clear that this should be the invariance group referred to in part 4. The

[1] in his seminar given at "Moonshine—the First Quarter Century and Beyond" in Edinburgh, July 2004.

isomorphism type of Δ is easily determined from the net, see Proposition 34 on page 99.

(6.2.3) Monstralisers

We define monstralisers as in [36, 38]: If $G \leqslant \mathbb{M}$, then the monstraliser of G is simply $C_{\mathbb{M}}(G)$. In general, $C_{\mathbb{M}}(G)$ will centralise a group $C_{\mathbb{M}}(C_{\mathbb{M}}(G)) = \overline{G} \geqslant G$, and so $\overline{G}$ is called the closure of G. Note that $\overline{G}$ is its own closure, and is the monstraliser of $C_{\mathbb{M}}(G)$. The pair of subgroups $C_{\mathbb{M}}(G)$ and $\overline{G}$ is called a monstraliser pair.

Given a net $\mathcal{N}(a,b,c)$ we may determine the net group $\langle a,b,c\rangle$ and its monstraliser, and we can use this information to classify nets.

(6.3) Finding Nets

Suppose that $\mathcal{N}(a,b,c)$ is (has its net group) centralised by an element g of prime order p. Then $a,b,c \in C_{\mathbb{M}}(g)$ and this group is generally much easier to work in than $\mathbb{M}$. Indeed, for $p \geqslant 11$ the subgroup $C_{\mathbb{M}}(g)$ is easy to work in and so the calculations are straightforward.

Our general strategy is first to obtain a list N of representatives for conjugacy classes of triples of involutions in $C_{\mathbb{M}}(g)$. We reduce N to a list in which each triple is a generator for one of the $C_{\mathbb{M}}(g)$-conjugacy classes of nets. To do this we work through N and for a chosen triple of involutions (a,b,c), use the braiding operations s and t to construct the net $\mathcal{N}(a,b,c)$. For each flag in $\mathcal{N}(a,b,c)$ take a representative triple (a',b',c') of the flag. We can now remove from N the representative for the conjugacy class of (a',b',c'), provided it occurs (strictly) after (a,b,c).

As we will be working in a subgroup of $\mathbb{M}$, many of the classes with representatives listed in N will fuse in $\mathbb{M}$. We can work out all of the possibilities by collecting together nets with properties invariant under conjugation. For example, nets with graph-isomorphic polyhedra *could* be conjugate.

(6.3.1) Finding Nets Centralised By An Element Of Prime Order At Least 7

From the Atlas [8] we find that $N_{\mathbb{M}}(\langle g\rangle)$ for g of prime order $p \geqslant 7$ is one of the following:

$$
\begin{array}{lll}
N(47A) = 47{:}23 \times 2 & N(31A) = 31{:}15 \times \mathrm{S}_3 & N(23A) = 23{:}11 \times \mathrm{S}_4 \\
N(19A) = (19{:}9 \times \mathrm{A}_5){:}2 & N(17A) = (17{:}8 \times \mathrm{L}_3(2))^{\cdot}2 & N(13A) = (13{:}6 \times \mathrm{L}_3(3))^{\cdot}2 \\
N(13B) = 13_+^{1+2}{:}(3 \times 4^{\cdot}\mathrm{S}_4) & N(11A) = (11{:}5 \times \mathrm{M}_{12}){:}2 & N(7A) = (7{:}3 \times \mathrm{He}){:}2 \\
N(7B) = 7_+^{1+4}{:}(3 \times 2^{\cdot}\mathrm{S}_7^-) & &
\end{array}
$$

We can omit the cases $N(13B)$ and $N(7B)$ since neither of these contain $2A$ involutions—no element of order 14 squares to an element in class $7B$ and has 7th power in class $2A$ so no involution centralises a $7B$ element. Similarly, no element of order 26 squares to class $13B$ and has 13th power in class $2A$. In the remaining cases the centralisers are easy to work in and so there will be no computational trouble. We now describe our computer programs, and a naïve net finding program that is sufficient for these cases.

(6.3.2) Net Functions: Data Structures And Supporting Functions

Our net finding programs rely on a number of basic net functions, in the Magma program `NetFunctions3.mag`.

We define two record formats:

1. A net `n` will be stored in a `netRF` variable which contains:
   (a) A generating triple, `n.rep`.
   (b) A number, `n.ismade`. This is 0 if the net is not made, 1 if the net has been made, and 2 if the net is to be ignored. (If we find the net to be conjugate to another net then we set this flag to 2 instead of deleting the net and re-ordering the array of nets.)
   (c) A list `n.flags` of the flags of the net, each is a `netFlagRF` variable (defined below).
   (d) A list `n.verts` of the vertices. Each vertex is a list of the indices of the flags it contains, stored in the order of applying s (*i.e.,* anti-clockwise).
   (e) A list `n.edges` of the edges as pairs (or singletons) of vertices.
   (f) A list `n.faces` of the faces, stored as a list indices of `n.flags` in the order of application of

x (*i.e.*, anti-clockwise).

(g) The modulus of the net, `n.m`.

(h) The order of the net, `n.ord`.

2. A net flag `f` is a variable of type `netFlagRF` which contains:

   (a) A representative triple, `f.trip`.

   (b) The indices of the flags in the same vertex as `f`, `f.v`.

   (c) The indices of the flags in the same edge as `f`, `f.e`.

   (d) The order of the face that contains the flag, `f.n`.

All of the net functions are described in detail below. The most important net function `netMake` takes a triple (a, b, c) and returns a completed `netRF` record.

`slp2net`: A triple of involutions (a, b, c) may be stored as a triple of straight line programs (w_a, w_b, w_c) in order to save space. This function takes 2 arguments, the first is a triple of SLPs and the second is a homomorphism f from the SLP group to the matrix or permutation group G in which we are working. Note that the triple of involutions is

$$(f(w_a), f(w_a)^{f(w_b)}, f(w_a)^{f(w_c)}) \tag{32}$$

`netVert`: Take a triple (a, b, c) of involutions and apply the braiding operation s to return (b, c, a^{bc}).

`netEdge`: Take a triple (a, b, c) of involutions and apply the braiding operation t to return (c, b^c, a^{bc}).

`netFace`: Take a triple (a, b, c) of involutions and apply the braiding operation x to return (b, a^b, c).

`netFlagFind`: Given a triple T and an array L of flags the function `netFlagFind` returns i if $T \in L[i]$ and returns $|L| + 1$ if $T \notin L[i]$ for all i. To do this each flag representative (a, b, c) is used to produce a list of all the triples in the flag *i.e.*, $(a, b, c)^{(abc)^k}$ for $1 \leqslant k \leqslant |abc|$. Every triple in the flag is then compared with T.

This function is used to construct nets, and accounts for most of the running time of `netMake`. Comparing the values of the `n` components of the flag records provides a quick test for non-equality of flags.

`array2graph`: Convert an adjacency matrix into a graph.

`netMake`: Take a `netRF` record, `net` and complete the fields. Initialise `net.flags = []` and

```
net.flags[1] := rec<netFlagRF | t:= net.rep, v:=[1], e:=[1],
                                n:=Order(net.rep[1]*net.rep[2])>
```

Let `t := netEdge(net.flags[i].t)` and `n := netFlagFind(net.flags, t)`. If `t` corresponds to a new flag then we add a new `netFlagRF` variable to `net.flags` with component `e := [i,n]`. We must also add `n` to `net.flags[i].e`. Thus the components `e` record which flags form edges of the net. If the flag with representative `t` has been seen before then we simply update the `e` components of the two records.

Now let `t := netVert(net.flags[i].t)`, then we use `netFlagFind` as above but this time update the `v` components of the `netFlagRF` records. If `t` is not in the flag `net.flags[i]` then it is in an un-collapsed vertex and so we know that `netVert(t)` is a third distinct flag so we process it with `netFlagFind`.

Having applied `netEdge` and `netVert` to `net.flags[i].t` we increment `i`, thus constructing the net by a breadth first search of the flags.

To find the faces we use `netFace` and `netFlagFind` as above and fill in the `f` components of the `netFlagRF` records. The component `net.verts` is an array of the vertices of the net: Each entry is an array of the indices in `net.flags` of the flags in the vertex. We can then use the `e` components of the flags to write the edges as pairs of indices of `net.verts`, this is `net.edges`. We leave the faces, `net.faces`, as lists of orbits on flags.

`netAreConj`: Test whether two nets, `net1` and `net2` *could* be conjugate in $\mathbb{M}$. To do this we compare invariants of the nets, if we find any that differ then the nets cannot be conjugate. The invariants are:

(i) The orders of the nets.

(ii) Whether each net has the same number of faces of each shape.

(iii) The moduli of the nets.

(iv) The group isomorphism type of the net groups.

(v) Graph isomorphism of the net polyhedra, Γ_1 and Γ_2. To test this we first remove double edges, loops, and collapsed edges as Magma graphs do not support these features. If the resulting graphs Γ_1' and Γ_2' are isomorphic with isomorphism ϕ then we obtain all possible isomorphisms by composition of ϕ with an automorphism of Γ_2'. We find which of these isomorphisms, if any, extend to isomorphisms from Γ_1 to Γ_2. Note that the extension must also preserve orientation of the faces. If none extend then the nets cannot be conjugate.

`netGenus`: Returns 1 if all faces have six sides, otherwise returns 0.

(6.3.3) Simple Net Finding Program

Our simplest program is `NetFinder3.mag` which works well for $p \geqslant 7$. First of all we wish to find a list N of representatives for the conjugacy classes of triples of involutions in a group K. We actually work up to conjugacy in an upward extension G of K (typically $G = K.2$). The group G need not be a subgroup of $\mathbb{M}$, for example when $p = 17$ we have $K = L_3(2)$ but there is no extension to $L_3(2){:}2$ in $\mathbb{M}$. There is, however, an element of order 16 in $\mathbb{M}$ which acts as the outer automorphism of $L_3(2)$. We can therefore work in $L_3(2){:}2$ to reduce the number of conjugacy classes of triples we need to consider. In all our cases only one conjugacy class of G fuses to class $2A$ of $\mathbb{M}$.

Let $G = \langle g_1, g_2 \rangle$ and let $g \in G$ be a representative for the conjugacy class of G that fuses to $2A$ in $\mathbb{M}$. Let $H = C_G(g)$ and let $\phi\colon G \to P$ be the permutation representation of G on the cosets of H. Let $p_k = \phi(g_k)$ for $k \in \{1, 2\}$. We have $H \cong P_1 = \mathrm{stab}_P(1)$. Let X be a set of orbit representatives for the action of P_1 and for each $x_i \in X$ let q_i be an element of P that takes 1 to x_i. Let $P_i = \mathrm{stab}_{P_1}(x_i)$ ($x_1 = 1$ so this notation is consistent) and let Y_i be a list of orbit representatives for P_i. For each $y_{ij} \in Y_i$ let $r_{ij} \in P$ take 1 to y_{ij}. The list of representatives of conjugacy classes of triples is

$$N = \{(g, g^{\phi^{-1}(q_i)}, g^{\phi^{-1}(r_{ij})}) \mid i \in X, j \in Y_i\}$$

where we obtain $\phi^{-1}\colon P \to G$ by mapping p_k to g_k for $k \in \{1, 2\}$.

In practice we can save some time and memory by not asking Magma for ϕ. To obtain $\phi^{-1}(r_{ij})$ we evaluate `ActingWord(P,1,`r_{ij}`)` in G and this leads use to store triples in N as

$$(1,\texttt{ActingWord(P,1,}q_i\texttt{)},\texttt{ActingWord(P,1,}r_{ij}\texttt{)})$$

and hence equation (32). This saves memory when group elements are comparatively large, *e.g.,* for He:2 when $p = 7$.

We now reduce N to a list of representative generating triples for the conjugacy classes of nets in G. We work along the list N and for a triple (a, b, c) we use `netMake` to create the net $\mathcal{N}(a, b, c)$. Let (d, e, f) be a representative for one of the flags of $\mathcal{N}(a, b, c)$. Then (d, e, f) is conjugate to some element (a, b', c') of N. This must occur after (or be equal to) (a, b, c) for otherwise (a, b, c) would have been removed from N when we examined $\mathcal{N}(a, b', c')$.

The conjugacy testing begins with `IsConjugate(G,a,d)` which of course returns `true`. It also returns an element of G that conjugates (d, e, f) to (a, e', f'). Next we require an element of $H = C_G(a)$ that conjugates (a, e', f') to (a, b', f''), if such an element exists. If so then we must find an element of $C_G(a, b')$ that conjugates f'' to c', if one exists. But $C_G(a, b') = \phi^{-1}(\text{stab}_{P_1}(x_i))$ so we do not need to use the `Centraliser` function.

Finally, we use `netAreConj` to find which of the remaining triples generate nets that could be conjugate in $\mathbb{M}$, even though they are not conjugate in G.

After some work optimising the programs, we could perform the entire procedure with $G = \mathrm{M}_{12}$:2 in under 10 seconds.

(6.3.4) Results for p at least 11

The programs described in the previous section have no problem in finding representatives for (at least) all the nets centralised by an element of prime order $p \geqslant 11$. In particular we find:

(i) For $p = 47$ there is no work to do: There is only the "trivial" net, $\mathcal{N}(a, a, a)$.

(ii) For $p = 31$ we work in $G = K = \mathrm{S}_3$ and find 5 conjugacy classes of flags which produce 2 nets. These two nets have different polyhedra and so cannot be conjugate in $\mathbb{M}$.

(iii) For $p = 23$ we work in $G = K = \mathrm{S}_4$ and find 12 conjugacy classes of flags which produce 4 nets, each with a different polyhedron *i.e.,* no two of these nets are conjugate in $\mathbb{M}$.

(iv) For $p = 19$ we work in $G = S_5$ for $K = A_5$ and find 34 conjugacy classes of flags which produce 7 nets. No two of these nets can be conjugate in $\mathbb{M}$.

(v) For $p = 17$ we work in $G = L_3(2).2$ for $K = L_3(2)$ and find 38 conjugacy classes of flags which produce 9 nets. No two of these nets can be conjugate in $\mathbb{M}$.

(vi) For $p = 13$ we work in $G = L_3(3).2$ for $K = L_3(3)$ and find 188 conjugacy classes of flags which produce 25 nets. Of these nets, there are two pairs of nets that could be conjugate in $\mathbb{M}$.

(vii) For $p = 11$ we work in M_{12}:2 for $K = M_{12}$ and find 452 conjugacy classes of flags which produce 44 nets. Of these nets there are two pairs and one triple of nets that could be conjugate in $\mathbb{M}$.

The results for $p \geqslant 11$ are summarised in the following table. The first column gives a description of the net corresponding to its location on the CD. Net p.q.[r] is `nets[r]` in `p/results.mag`. The columns V, E, and F give the number of vertices, edges, and faces (respectively) of the net. "Order" means the order of the net, the "$|G|$" column gives the order of the net group, this column is used to sort the table. The seventh column gives the modulus of the net. The remaining columns give the group, its monstraliser, and the net number as it appears in [39] and [41]. The net number nX, m is Norton's name for net m of class nX. An asterisk and number in the final column refers to the notes following the table, and a bullet denotes a net not previously named.

Index	V	E	F	Order	Mod.	$\|G\|$	Group	Monstraliser	Quotient of Γ	NNN
11:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$	1	2a,1
13:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$	1	2a,1
17:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$	1	2a,1
19:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$	1	2a,1
23:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$	1	2a,1
31:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$	1	2a,1
11:22:[38]	2	3	3	1	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	1a,1
13:2:[2]	1	2	2	2	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	2a,2
13:6:[9]	2	3	3	1	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	1a,1
17:2:[2]	1	2	2	2	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	2a,2
17:5:[7]	2	3	3	1	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	1a,1
19:2:[2]	1	2	2	2	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	2a,2
19:5:[7]	2	3	3	1	1	4	2^2	$2^2.{}^2E_6(2)$	S_3	1a,1
11:2:[2,3]	1	2	2	2	1	4	2^2	$2^{2+22}.Co_2$	S_3	2a,3∗1
23:2:[2]	1	2	2	2	1	4	2^2	$2^{2+22}.Co_2$	S_3	2a, 3
11:3:[4]	2	2	2	2	2	6	S_3	Fi_{23}	2^2:3	2a,4
13:3:[3,4]	2	2	2	2	2	6	S_3	Fi_{23}	2^2:3	2a, 4 ∗2
17:3:[3]	2	2	2	2	2	6	S_3	Fi_{23}	2^2:3	2a,4
23:3:[3]	2	2	2	2	2	6	S_3	Fi_{23}	2^2:3	2a,4

continued...

...continued

Index	V	E	F	Order	Mod.	$\|G\|$	Group	Monstraliser	Quotient of Γ	NNN
19:3:[3]	2	2	2	2	2	6	S_3	Th	$2^2{:}3$	2a,5
31:2:[2]	2	2	2	2	2	6	S_3	Th	$2^2{:}3$	2a,5
11:7:[10,11,41]	2	3	3	2	1	8	2^3	see note	S_3	$*3$
13:4:[5]	2	3	3	2	2	8	D_8	$2{\cdot}F_4(2)$	$2^2{:}S_3$	2a,7
17:4:[4]	2	3	3	2	2	8	D_8	$2{\cdot}F_4(2)$	$2^2{:}S_3$	2a,7
13:7:[10]	2	3	3	4	2	8	D_8	$2{\cdot}F_4(2)$	S_3	4b,1
17:6:[9]	2	3	3	4	2	8	D_8	$2{\cdot}F_4(2)$	S_3	4b,1
11:4:[5]	2	3	3	2	2	8	D_8	$2^{1+22}.McL$	$2^2{:}S_3$	2a,6
11:5:[6]	4	6	4	2	2	10	D_{10}	HN	A_5	2a,8
19:4:[4]	4	6	4	2	2	10	D_{10}	HN	A_5	2a,8
11:6:[7]	4	6	4	2	2	12	D_{12}	$2{\cdot}Fi_{22}$	$(2^2{:}3 \times 3){:}2$	2a,9
13:5:[6]	4	6	4	2	2	12	D_{12}	$2{\cdot}Fi_{22}$	$(2^2{:}3 \times 3){:}2$	2a,9
11:24:[47]	2	3	3	3	3	12	D_{12}	$2{\cdot}Fi_{22}$	S_3	3a,1
13:8:[13]	2	3	3	3	3	12	D_{12}	$2{\cdot}Fi_{22}$	S_3	3a,1
13:9:[14]	1	2	2	6	3	12	D_{12}	$2{\cdot}Fi_{22}$	S_3	6a,1
11:8:[12,43]	1	2	2	6	3	12	D_{12}	see note	S_3	$*4$
11:23:[46]	4	6	4	2	2	16	$2 \times D_8$	$2^{1+22}.M_{22}$	$2^2{:}S_3$	2b,4
11:11:[17]	2	3	3	4	2	16	$2 \times D_8$	$2^{1+22}.M_{22}$	S_3	4b,3
13:20:[50]	4	6	4	2	2	18	$3^2{:}2$	$O_8^+(3)$	$2^2{:}3$	2a,12
11:10:[16]	1	2	2	10	5	20	$2 \times D_{10}$	$2{\cdot}HS.2$	S_3	$\bullet$ $*5$
17:7:[11]	2	2	2	4	4	24	S_4	$S_8(2)$	$2^2{:}3$	4b,2
17:8:[12]	3	5	3	3	3	24	S_4	$S_8(2)$	$3^3{:}S_4$	3a,2
11:15:[25]	2	2	2	4	4	24	S_4	${}^3D_4(2)$	$2^2{:}3$	4b,3
13:11:[20]	2	2	2	4	4	24	S_4	${}^3D_4(2)$	$2^2{:}3$	3c,1
13:10:[19]	3	5	3	3	3	24	S_4	${}^3D_4(2)$	$3^3{:}S_4$	3c,1
23:4:[6]	2	2	2	4	4	24	S_4	$2^{11}.M_{23}$	$2^2{:}3$	4a,4
11:25:[49]	8	12	6	2	2	24	$2^2 \times S_3$	$2^2.U_6(2)$	$(2^2{:}3 \times 3){:}2$	2b,7
11:12:[19]	2	3	3	6	3	24	$2^2 \times S_3$	$2^2.U_6(2)$	S_3	6a,3
13:12:[22]	4	6	4	6	6	36	$3^2{:}2^2 \cong S_3 \times S_3$	$O_7(3)$	$(2^2{:}3 \times 3){:}2$	6a,9
13:15:[26]	6	9	5	3	3	36	$3^2{:}2^2 \cong S_3 \times S_3$	$O_7(3)$	$3_+^{1+2}{:}2$	3a,4
13:19:[40]	6	8	4	8	4	48	$2{\cdot}S_4$	$2 \times {}^2F_4(2)'$	$2^6{\cdot}((2^2{:}3 \times 2^2{:}3){:}2)$	$\bullet$
11:17:[27]	4	6	4	4	4	48	$2 \times S_4$	$2^{11}.M_{22}$	$(2^2{:}3 \times 3){:}2$	4a,5
11:28:[59]	6	9	5	3	3	48	$2 \times S_4$	$2^{11}.M_{22}$	$3^3{:}S_4$	3a,5
11:29:[61]	3	5	3	6	3	48	$2 \times S_4$	$2^{11}.M_{22}$	$3^3{:}S_4$	6a,4
13:18:[34]	4	6	4	6	2	54	$3_+^{1+2}{:}2$	$G_2(3)$	$2^2{:}3$	6a,12$*6$
19:7:[10]	6	9	5	3	3	60	A_5	$U_3(8).3_1$	$2^9{:}A_9$	3c,2
19:6:[8]	4	5	3	5	5	60	A_5	$U_3(8).3_1$	S_{10}	5a,2
11:33:[70]	6	9	5	3	3	60	A_5	A_{12}	$2^9{:}A_9$	3a,3
11:34:[71]	4	5	3	5	5	60	A_5	A_{12}	S_{10}	5a,1
13:17:[29]	2	3	3	6	6	72	$3^2{:}D_8$	$L_4(3).2_2$	$2^2{:}S_3$	6a,6
13:14:[25]	8	12	6	4	4	72	$3^2{:}D_8$	$L_4(3).2_2$	$2^6{:}3_+^{1+2}{:}2$	4b,20
11:39:[116]	2	2	2	8	8	96	$4^2{:}D_6$	$2^{10}.M_{11}$	$2^2{:}3$	$\bullet$
13:16:[28]	18	27	11	3	3	108	$3^2{:}(2 \times S_3) \cong 3_+^{1+2}{:}2^2$	$G_2(3)$	$3{\cdot}3_+^{1+2}{:}2$	3c,3
13:13:[23]	4	6	4	6	6	108	$3^2{:}(2 \times S_3) \cong 3_+^{1+2}{:}2^2$	G_23	$(2^2{:}3 \times 3){:}2$	6a,10

continued...

...continued

Index	V	E	F	Order	Mod.	$\lvert G\rvert$	Group	Monstraliser	Quotient of Γ	NNN
11:9:[15]	6	9	5	6	3	120	$2 \times A_5$	$2{\cdot}M_{22}$	$2^9{:}A_9$	6a,23
11:13:[21]	4	5	3	10	5	120	$2 \times A_5$	$2{\cdot}M_{22}$	S_{10}	•
17:9:[21]	3	4	2	7	7	168	$L_3(2)$	$S_4(4){:}2$	S_7	•
11:19:[31]	4	6	4	8	8	192	$4^2{:}D_{12}$	$*7$	$(2^2{:}3 \times 3){:}2$	•
11:14:[24]	5	8	4	10	5	240	$2 \times S_5$	$2{\cdot}M_{22}$	$(A_5 \times A_5 \times A_5){:}S_4$	•
11:26:[55]	10	15	7	5	5	240	$2 \times S_5$	$2{\cdot}M_{22}$	$(A_5 \times A_5 \times A_5){:}S_4$	5a,7
11:16:[26]	6	9	5	6	6	240	$2 \times S_5$	$2{\cdot}M_{22}$	$(A_6 \times A_6 \times A_6){:}S_4$	6a,22
11:27:[57]	8	12	6	4	4	240	$2 \times S_5$	$2{\cdot}M_{22}$	$(A_8 \times A_8 \times A_8){:}S_3$	4a,15
13:21:[51,56]	6	8	4	8	8	432	$3^2{:}2{\cdot}S_4$	$L_3(3)$	$2^6{\cdot}((2^2{:}3 \times 2^2{:}3){:}2)$	•
11:31:[67]	10	15	7	5	5	660	$L_2(11)$	M_{12}	$2^{14}{:}S_{15}$	5a,9 or 5a,10
11:32:[69]	10	15	7	5	5	660	$L_2(11)$	M_{12}	$2^{14}{:}S_{15}$	5a,9 or 5a,10
11:30:[66]	12	18	8	6	6	660	$L_2(11)$	M_{12}	$2^{17}{:}A_{18}$	6a,28
11:35:[102]	8	11	5	11	11	660	$L_2(11)$	M_{12}	S_{22}	•
11:20:[32]	6	8	4	8	8	720	$M_{10} \cong A_6{\cdot}2_3$	M_{11}	A_{16}	• $*8$
11:21:[33]	4	5	3	10	10	720	$M_{10} \cong A_6{\cdot}2_3$	M_{11}	S_{10}	• $*8$
13:22:[76]	5	7	3	13	13	5616	$L_3(3)$	$3^2{:}2{\cdot}S_4$	A_{13}	•
13:23:[80]	5	7	3	13	13	5616	$L_3(3)$	$3^2{:}2{\cdot}S_4$	A_{13}	•
11:36:[103]	12	18	8	6	6	95040	M_{12}	$L_2(11)$	$2^{17}{:}A_{18}$	6b,8 or 6b,9
11:38:[112]	12	18	8	6	6	95040	M_{12}	$L_2(11)$	$2^{17}{:}A_{18}$	6b,8 or 6b,10
11:40:[118]	12	18	8	6	6	95040	M_{12}	$L_2(11)$	$2^{17}{:}A_{18}$	6b,8 or 6b,11
11:18:[29]	16	24	10	8	8	95040	M_{12}	$L_2(11)$	A_{48}	•
11:37:[109]	8	11	5	11	11	95040	M_{12}	$L_2(11)$	S_{22}	•

1. In 11:2:[2] and 11:2:[3] we have $a = b$ so the net group is $\{1, a, c, ac\}$. As ac is not conjugate to a in M_{12} it must be in $\mathbb{M}$ class $2B$, so the monstraliser of the net group is $2^{2+22}{\cdot}Co_2$ and both nets are 2a,3.

2. Considering centraliser orders in $L_3(3){:}2$, classes $3A$, $3B$ fuse to classes $3A$, $3C$ of $\mathbb{M}$ respectively. Therefore 13:3:[3] is also known as 2a,4 which is monstralised by Fi_{23} and 13:3:[4] is also known as 2a,5.

3. For net 11:7:[41] $abc \in 2B$ of $M_{12}{:}2$ which fuses to $2B$ of $\mathbb{M}$. Therefore 11:7:[41] is also known as 2b,1.

   The group of net 11:7:[10] contains a $2B$-pure subgroup 2^2 so its monstraliser must be a subgroup of $2^{2+11+22}.(S_3 \times M_{24})$ so this net must be 2a,11. As the net group of 11:7:[11] contains just one $2B$ involution it cannot be conjugate to 11:7:[10] and so this is net 2a,10.

4. The net group of 11:8:[43] contains $2B$ elements and is conjugate in $M_{12}{:}2$ to that of net 11:6:[7] and so must be centralised by $2{\cdot}Fi_{22}$. Hence this is net 6a,1. The net group of 11:8:[12] contains only $2A$

involutions thus cannot be conjugate to 11:8:[43] and must be net 6a,2.

5. Net 11:10:[16] has order 10 and so does not appear on Norton's list. The central involution in the net group has two complementary groups D_{10}, and one of them is conjugate in M_{12}:2 to the net group of net 11:5:[6] which is monstralised by HN. The central involution is in $\mathbb{M}$-class $2A$ from which it follows that the centraliser of net 11:10:[16] is 2·HS:2.

6. The net group 3_+^{1+2}:2 for net 13:18:[34] is a subgroup of 3^2:2·S_4 which forms a monstraliser pair with $G_2(3)$.

7. The net 11:19[31] has net group 4^2:D_{12}. From [38] we see that the only option is $4^2:(2 \times S_3) \times 2^{10}:L_2(11)$. Neither of the other two options contain the full D_{12}, in particular an involution that acts non-trivially on the 4^2.

8. The groups S_6:2 and M_{11} are a monstraliser pair, so the monstraliser of $A_6{\cdot}2_3$ is M_{11}.

(6.3.5) Results for $p = 7$

For $p = 7$ we work in He:2 using the permutation representation on 2058 points from the web Atlas. There are 2931 conjugacy classes of flags and these reduce to 190 conjugacy classes of nets. At least 144 of these are distinct $\mathbb{M}$-conjugacy classes of nets.

Index	V	E	F	Order	Mod.	$\|G\|$	Group	Monstraliser	Quotient of Γ	NNN
7:1:[1]	1	1	1	2	1	2	2	$2{\cdot}B$		2a,1
7:7:[14,29]	2	3	3	1	1	4	2^2			
7:2:[2,3,4]	1	2	2	2	1	4	2^2			
7:4:[7,8]	2	2	2	2	2	6	S_3			
7:9:[18,33,35,112,113]	2	3	3	2	1	8	2^3			
7:3:[5,6,10]	2	3	3	2	2	8	D_8			
7:8:[16]	2	3	3	4	2	8	D_8	$2.F_4(2)$		
7:5:[9]	4	6	4	2	2	10	D_{10}	HN		2a,8
7:6:[11]	4	6	4	2	2	12	D_{12}	$2{\cdot}Fi_{22}$		2a,9
7:19:[43]	2	3	3	3	3	12	D_{12}	$2{\cdot}Fi_{22}$		3a,1
7:18:[42,126]	1	2	2	6	3	12	D_{12}	$2{\cdot}Fi_{22}$		6a,2
7:16:[39,52,53,118]	4	6	4	2	2	16	$2^2{\cdot}2^2 \cong 2 \times D_8$			
7:136:[1089]	8	12	6	4	1	16	$4{\cdot}2^2$	$2^{1+21}:U_3(5)$		4a,17
7:17:[40,48,49,116,123]	2	3	3	4	2	16	$2 \times D_8$			
7:112:[565]	4	6	4	2	2	18	3^2:2			
7:46:[134]	1	2	2	10	5	20	D_{20}			
7:23:[64]	8	12	6	2	2	24	$2^2 \times S_3$	$2^2:U_3(5)$		2b,7
7:11:[20,36,56]	3	5	3	3	3	24	S_4	$S_8(2)$		3a,2

continued...

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Index	V	E	F	Order	Mod.	$\|G\|$	Group	Monstraliser	Quotient of Γ	NNN
7:10:[19,37,57,141,143]	2	2	2	4	4	24	S_4			
7:22:[63]	2	3	3	6	3	24	$2^2 \times S_3$	2^2:$U_3(5)$		2b,7
7:70:[215]	2	3	3	6	6	24	$2^2 \times S_3$			
7:14:[25,130,132,133]	4	6	4	4	2	32	2^4:2			
7:71:[225]	8	12	6	4	2	32	$(2 \times D_8)$:2			
7:29:[77]	6	9	5	3	3	36	$S_3 \times S_3$			
7:26:[73]	4	6	4	6	6	36	$S_3 \times S_3$			
7:25:[72]	6	9	5	3	3	48	$2 \times S_4$	2^{11}:M_{22}		3a,5
7:28:[76,84]	4	6	4	4	4	48	$2 \times S_4$			
7:30:[79]	6	9	5	6	3	48	$2 \times S_4$			
7:31:[81,129,138,153]	3	5	3	6	3	48	$2 \times S_4$			
7:50:[152]	2	3	3	12	6	48	D_8:S_3			
7:94:[434]	4	6	4	6	2	54	3_+^{1+2}:2			
7:13:[24,69]	6	9	5	3	3	60	A_5	A_{12}		3a,3
7:12:[22,70]	4	5	3	5	5	60	A_5	$U_3(8)$:3		3c,2
7:24:[66,67,137]	4	6	4	4	4	64	2^4:2^2			
7:119:[627]	4	6	4	4	4	72	2^2:$(3^2$:2)			
7:53:[162]	6	9	5	6	3	72	D_{12}:D_{16}			
7:57:[166]	2	3	3	6	6	72	$(D_6 \times D_6)$:2			
7:114:[593]	6	9	5	3	3	96				
7:32:[86]	8	12	6	4	4	96				
7:27:[75]	6	9	5	6	3	96				
7:43:[103]	18	27	11	3	3	108				
7:37:[93]	4	6	4	6	6	108				
7:20:[54]	8	12	6	4	4	120	S_5			
7:21:[58]	5	8	4	5	5	120	S_5			
7:52:[160]	6	9	5	6	3	120	$2 \times A_5$			
7:15:[26]	6	9	5	6	6	120	S_5			
7:49:[149]	4	5	3	10	5	120	$2 \times A_5$			
7:33:[87]	8	12	6	4	4	144				
7:41:[98]	4	6	4	6	6	144				
7:54:[163]	4	6	4	12	12	144				
7:48:[148]	8	12	6	4	4	160				
7:69:[197,199]	3	4	2	7	7	168				
7:42:[100,170]	8	12	6	4	4	192				
7:35:[89]	6	9	5	6	6	192				
7:121:[644]	6	8	4	8	8	192				
7:67:[185]	2	3	3	10	10	200				
7:93:[423]	4	6	4	12	4	216				
7:36:[90]	8	12	6	4	4	240	$2 \times S_5$			
7:34:[88]	10	15	7	5	5	240				
7:40:[97]	6	9	5	6	6	240	$2 \times S_5$			
7:51:[154]	10	15	7	10	5	240				
7:59:[169]	5	8	4	10	5	240	$2 \times S_5$			

continued...

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Index	V	E	F	Order	Mod.	$\lvert G\rvert$	Group	Monstraliser	Quotient of Γ	NNN
7:130:[704]	8	12	6	4	4	288				
7:47:[147]	5	8	4	10	5	320				
7:95:[461,566]	3	4	2	14	7	336				
7:81:[253]	5	8	4	15	15	360	$(3 \times A_5){:}2$			
7:55:[164]	16	24	10	4	4	384				
7:83:[263]	6	9	5	6	6	384				
7:77:[243]	6	9	5	12	3	384				
7:142:[1496,1499]	8	12	6	4	4	400				
7:60:[172]	4	6	4	12	12	432	$2^2{:}(3^{1+2}{:}2^2)$			
7:76:[240]	8	12	6	8	4	512	$(2^6) : D_8$			
7:44:[105]	12	18	8	6	6	576	$(2^4) : 3 : D_{12}$			
7:45:[106]	12	18	8	6	6	576	$2^4{:}3{:}D_{12}$			
7:137:[1143]	14	21	9	7	7	672	$2^2{:}L_2(7)$			
7:118:[602]	7	11	5	14	7	672	$2^2{:}L_2(7)$			
7:117:[601]	8	12	6	4	4	720	$2 \times A_6$			
7:115:[595]	5	8	4	10	5	720	$2 \times A_6$			
7:56:[165]	6	9	5	12	6	768	$2^6{:}D_{12}$			
7:75:[231]	8	12	6	12	4	1080	$3{\cdot}A_6$			
7:72:[226,227]	5	8	4	15	5	1080	$3{\cdot}A_6$			
7:129:[702]	16	24	10	4	4	1152	$2^6{:}(3^2{:}2)$			
7:38:[95]	10	15	7	5	5	1440	$2 \times S_6$			
7:39:[96]	12	18	8	6	6	1440	$2 \times S_6$			
7:61:[175]	10	15	7	10	5	1440	$2 \times S_6$			
7:97:[473,474]	6	8	4	8	8	1728				
7:58:[167]	12	18	8	6	6	2160	$3{\cdot}S_6$			
7:65:[182]	12	18	8	12	12	2304	$2^6{:}(S_3 \times S_3)$			
7:125:[681]	16	24	10	8	8	4608	$2^6{:}(2^2{:}(3^2{:}2))$			
7:127:[691]	16	24	10	8	8	4608	$2^6{:}(2^2{:}(3^2{:}2))$			
7:63:[180]	36	54	20	6	6	6912	$2^6{:}(3^{1+2}{:}2^2)$			
7:101:[478]	4	6	4	12	12	6912	$2^6{:}(3^{1+2}{:}2^2)$			
7:134:[916]	10	15	7	5	5	7200	$(A_5 \times A_5){:}2$			
7:122:[677]	2	3	3	15	15	7200	$(A_5 \times A_5){:}2$			
7:73:[229]	7	11	5	21	7	7560	$3{\cdot}A_7$			
7:74:[230]	7	11	5	21	7	7560	$3{\cdot}A_7$			
7:78:[247,258]	8	12	6	8	8	7680	$2^6{:}S_5$			
7:79:[249,252]	10	15	7	10	10	7680	$2^6{:}S_5$			
7:62:[177]	12	18	8	12	6	7680	$2^6{:}S_5$			
7:139:[1387]	12	18	8	6	6	14400	$(A_5 \times A_5){:}2^2$			
7:68:[186]	10	15	7	10	10	14400	$(A_5 \times A_5){:}2^2$			
7:88:[279]	10	15	7	10	10	14400	$(A_5 \times A_5){:}2^2$			
7:141:[1434]	20	30	12	10	10	14400	$(A_5 \times A_5){:}2^2$			
7:66:[184]	8	12	6	12	12	14400	$(A_5 \times A_5){:}2^2$			
7:143:[1497]	4	6	4	12	12	14400	$(A_5 \times A_5){:}2^2$			
7:140:[1428]	10	15	7	15	15	14400	$(A_5 \times A_5){:}2^2$			

continued...

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Index	V	E	F	Order	Mod.	$\|G\|$	Group	Monstraliser	Quotient of Γ	NNN
7:85:[272,276]	15	23	9	15	15	23040	$2^6{:}((3 \times A_5){:}2)$			
7:113:[587]	10	15	7	5	5	80640	$2^2{\cdot}L_3(4)$			
7:133:[815]	14	21	9	7	7	80640	$2^2{\cdot}L_3(4)$			
7:116:[596]	7	11	5	14	7	80640	$2^2{\cdot}L_3(4)$			
7:89:[280]	12	18	8	12	12	138240	$2^6{:}3{\cdot}S_6$			
7:120:[631]	16	24	10	8	8	161280	$2^2{\cdot}L_3(4){:}2$			
7:80:[250]	14	21	9	14	14	161280	$2^2{\cdot}L_3(4){:}2$			
7:82:[257]	14	21	9	14	14	161280	$2^2{\cdot}L_3(4){:}2$			
7:84:[269]	14	21	9	28	14	161280	$2^2{\cdot}L_3(4){:}2$			
7:106:[486]	10	14	6	14	14	483840	$2^2{\cdot}L_3(4){:}S_3$			
7:111:[509]	10	15	7	15	15	483840	$2^2{\cdot}L_3(4){:}S_3$			
7:92:[286]	21	32	12	21	21	483840	$2^2{\cdot}L_3(4){:}S_3$			
7:99:[476]	10	14	6	28	28	483840	$2^2{\cdot}L_3(4){:}S_3$			
7:123:[679]	16	24	10	8	8	1958400	$S_4(2)$	$L_2(7)$		
7:124:[680]	16	24	10	8	8	1958400	$S_4(2)$	$L_2(7)$		
7:128:[699]	16	24	10	8	8	1958400	$S_4(2)$	$L_2(7)$		
7:132:[708]	16	24	10	8	8	1958400	$S_4(2)$	$L_2(7)$		
7:138:[1377,1392]	16	24	10	8	8	1958400	$S_4(2)$	$L_2(7)$		
7:126:[685]	20	30	12	10	10	1958400	$S_4(4){:}2$	$L_2(7)$		
7:131:[707]	20	30	12	10	10	1958400	$S_4(4){:}2$	$L_2(7)$		
7:64:[181]	24	36	14	12	12	1958400	$S_4(4){:}2$	$L_2(7)$		
7:86:[274]	15	23	9	15	15	1958400	$S_4(4){:}2$	$L_2(7)$		
7:90:[283]	17	26	10	17	17	1958400	$S_4(4){:}2$	$L_2(7)$		
7:91:[285]	17	26	10	17	17	1958400	$S_4(4){:}2$	$L_2(7)$		
7:144:[2508,2929]	98	147	49	7	7	4030387200	He	7:3		
7:96:[472]	24	36	14	12	12	4030387200	He	7:3		
7:105:[485]	24	36	14	12	12	4030387200	He	7:3		
7:102:[479]	10	14	6	14	14	4030387200	He	7:3		
7:104:[484]	10	14	6	14	14	4030387200	He	7:3		
7:108:[493]	10	14	6	14	14	4030387200	He	7:3		
7:135:[922]	28	42	16	14	14	4030387200	He	7:3		
7:98:[475]	12	17	7	17	17	4030387200	He	7:3		
7:103:[481]	12	17	7	17	17	4030387200	He	7:3		
7:109:[495]	12	17	7	17	17	4030387200	He	7:3		
7:87:[275]	21	32	12	21	21	4030387200	He	7:3		
7:107:[489]	14	21	9	21	21	4030387200	He	7:3		
7:110:[502]	14	21	9	21	21	4030387200	He	7:3		
7:100:[477]	10	14	6	28	28	4030387200	He	7:3		

The net groups of 7:9:[18,33,35,112,113] are all 2^3 and these are easy to classify using the information in Norton's list [39]:

- The class of 7:9:[112] is $2A$ and the net group contains a $2B$-pure 2^2 which is monstralised by

$2^{2+11+22}$:($S_3 \times M_{24}$). This group cannot contain $U_6(2)$ so from [39] clearly this net must be 2a,11.

- Net 7:9:[33] is the only other in this collection that has class $2A$. It contains no $2B$-pure subgroup, so is 2a,10.
- Net 7:9:[35] contains four $2A$-pure 2^2 subgroups and just one element of class $2B$. Now, ${}^2E_6(2)$ monstralises a $2A$-pure 2^2 and the centraliser in this group of another $2A$ element is 2^{1+20}:$U_6(2)$ so clearly this net must be 2b,1.
- The net group of net 7:9:[113] contains three elements of class $2A$ and four of $2B$. This includes a $2B$-pure 2^2 subgroup and so this net must be 2b,3.
- The remaining net, 7:9:[18], must therefore be 2b,2.

In this list we find our first net of genus 1. The net group is He. This is also the largest net found yet, having 98 vertices, 147 edges, and 49 faces.

(6.4) A Presentation For The Stabiliser Of A Flag

Now that we have found some nets, we can at last give some examples to illustrate the many definitions given at the start of the chapter.

Figure 6 shows net 19:7:[10] "exploded"—the vertices have been expanded to show their flags. It is immediate from the diagram that there is a permutation representation of Γ on 18 points with

$$\begin{aligned} s &\mapsto (1\,2\,3)(4\,5\,6\,)(7\,8\,9)(10\,11\,12)(13\,14\,15)(16\,17\,18) \\ t &\mapsto (1\,4)(2\,7)(3\,5)(6\,10)(8\,13)(9\,16)(11\,18)(12\,14)(15\,17) \end{aligned} \tag{33}$$

Figure 6 is almost a coset diagram for Γ on the cosets of the kernel, N, of this representation: To make a coset diagram each red edge must be replaced with two edges, one in each direction.

We shall use Reidemeister-Schreier and the presentation $\Gamma = \langle s, t \mid s^3 = t^2 = 1 \rangle$ to find a presentation for the stabiliser in Γ of a flag. We select a spanning tree from the green and red edges (vertices and edges) as follows (the procedure is described in [20]):

- In an uncollapsed vertex delete one of the green edges (Figure 7(a)). The relation $s^3 = 1$ ensures that the generator for N obtained from this edge is trivial.
- In a collapsed vertex we obtain one generator of order 3 (Figure 7(b)).

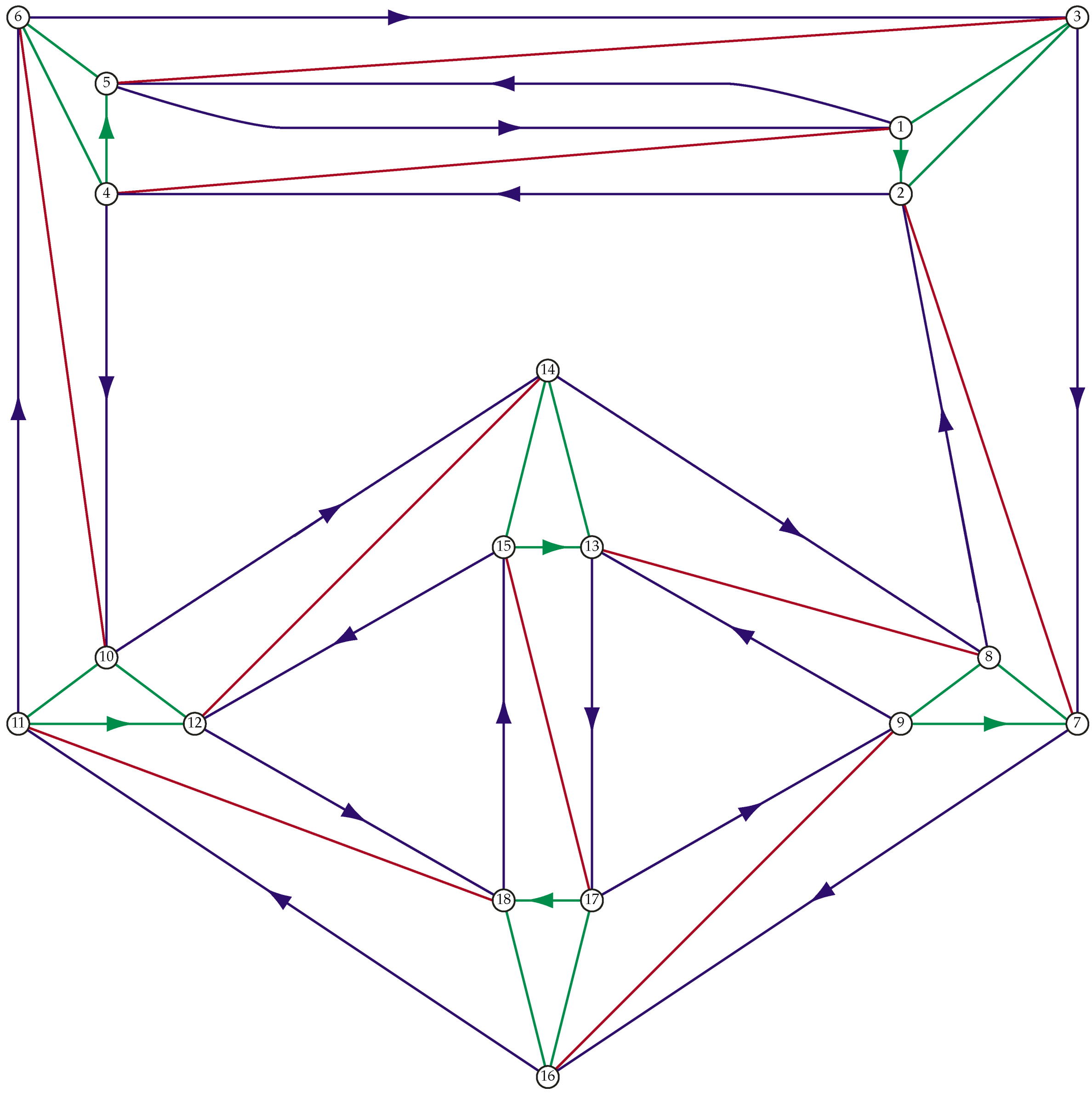

Figure 6: An "exploded" drawing of net 19:7:10. The vertices are the flags of the net and are numbered as on the CD. In green are s-orbits, the t-orbits are red, and the x-orbits are blue. Arrows indicate the direction of s and x. Since t-orbits have size 2, replacing each red edge with two edges—one in each direction—will produce a coset diagram for the action of Γ on the cosets of a flag stabiliser.

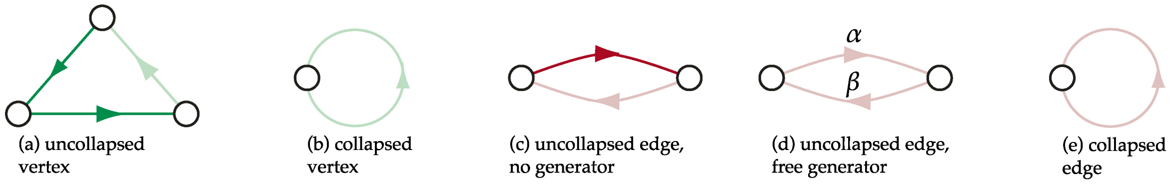

Figure 7: Choosing a spanning tree for use with the Reidemeister-Schreier algorithm.

- From each pair of red edges (each edge in the net) remove one red edge. The relation $t^2 = 1$ ensures that the generator obtained from the removed edge is trivial (Figure 7(c)).

- Remove additional red edges to produce a spanning tree. This creates a number of pairs of red edges not in the spanning tree (Figure 7(d)). These contribute two generators, say α and β and the relation $t^2 = 1$ means that $\alpha\beta = 1$. This is equivalent to one free generator, α, and one trivial generator, $\alpha\beta$.

- Collapsed edges do not form part of the spanning tree (Figure 7(e)) and so contribute a generator of order 2.

It is sufficient to look at the net, rather than the coset diagram to obtain the spanning tree, see Figure 8. In fact, the following proposition shows that the structure of the flag stabiliser is easily obtained from the net.

Proposition 34 (Theorem A.3.2 in [20]) *In a net of genus g with n flags, v collapsed vertices, e collapsed edges, and f faces the stabiliser in Γ of a flag is a group $3^{*v} * 2^{*e} * F$ where F is a free group of rank $2g + f - 1$.*

Proof. The quotient $3^{*v} * 2^{*e}$ is obvious from above. The rank of the free group is the number of edges of the net removed to make a spanning tree. A spanning tree has one less edge than it has vertices and the number of vertices is $v + \frac{n-v}{3}$. Similarly, the number of edges is $\frac{n-e}{2}$ and so the number of removed edges is

$$\begin{aligned}\frac{n-e}{2} - \left(v + \frac{n-v}{3} - 1\right) &= -\frac{e}{2} - \frac{2v}{3} + \frac{n}{6} + 1 \\ &= 2g + f - 1 \quad \text{by equation (29)}\end{aligned}$$ □

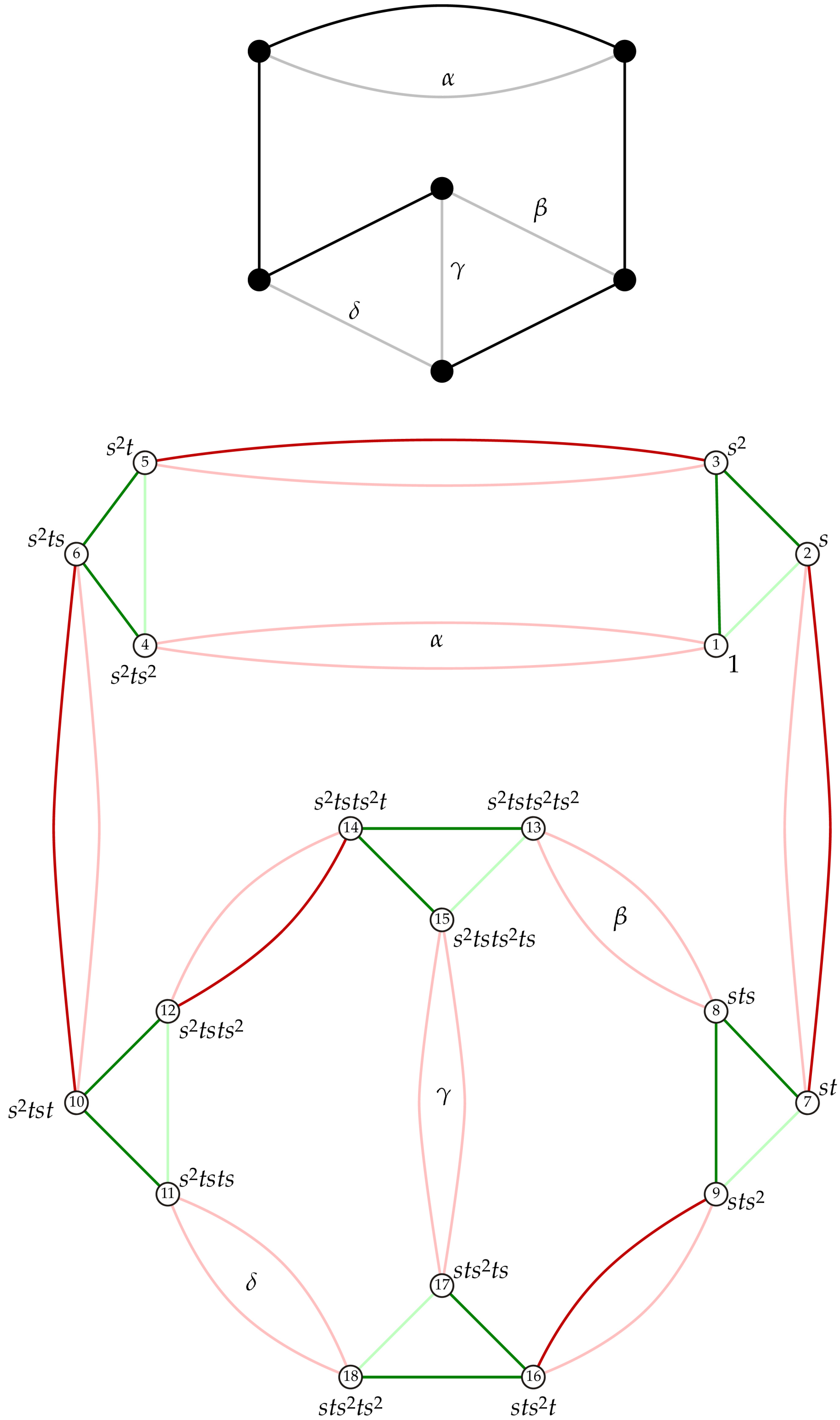

Figure 8: A spanning tree for the net determines the spanning tree for the coset diagram. (Net 19:7:10 is shown again.) Generators for the flag stabiliser are $\alpha = tsts$, $\beta = s^2tsts^2ts^2t^2t^2$, $\gamma = st^2tst^2tst^2ts$, and $\delta = st^2t^2t^2t^2ts$.

(6.5) Nets Centralised By An Element Of Class 5B

If $g \in 5B$ then $N_{\mathbb{M}}(\langle g \rangle) = 5_+^{1+6}{:}(4{\cdot}J_2{\cdot}2)$. The complement of the O_5 subgroup may be more clearly expressed as $(4 \circ 2{\cdot}J_2){\cdot}2$ and we may obtain this group from $2{\cdot}J_2{\cdot}2$ by multiplying elements in the outer half of the group by an 8th root of unity. The nets centralised by an element of class $5B$ are easy to find with our program `NetFinder3`, but first we must construct the group and find representatives for the conjugacy classes of involutions that fuse to $2A$.

From [52] we obtain two 6-dimensional matrices, C' and D, with coefficients in $\mathbb{F}_{25}$ and that are standard generators for $2{\cdot}J_2{\cdot}2$. The generator C' has order 4, lies in the outer half of the group, and squares to the central involution. Thus if $\omega \in \mathbb{F}_{25}$ is an 8th root of unity then $C = \omega C'$ and D generate a group of shape $4{\cdot}J_2{\cdot}2$.

This representation is equivalent to one over $\mathbb{F}_5$. We use the Standard Basis algorithm (see Section 2.1.1) to re-write C and D over $\mathbb{F}_5$, giving C_5 and D_5 say. If C_5' and D_5' are generators for an equivalent representation over $\mathbb{F}_5$ then as usual we search elements of the $\mathbb{F}_5$-algebra generated by C_5' and D_5' for a matrix of nullity 1 and spin up the null vector $\mathbf{v}$ to form the standard basis. As we are working over $\mathbb{F}_5$ all coefficients of C_5' and D_5' written in the standard basis must be in $\mathbb{F}_5$. Therefore if we search only amongst those $\lambda C + \mu D$ for λ, μ in the prime subfield $\mathbb{F}_5$ then C and D expressed in the standard basis must have all coefficients in $\mathbb{F}_5$.

Now, suppose that S^{-1} is the matrix of a symplectic form, "$\langle \cdot, \cdot \rangle$" say, on 6-dimensional $\mathbb{F}_5$ space. As D_5 lies in $2{\cdot}J_2$ it must preserve the form, but C_5 conjugates $1 \neq z \in Z(5^{1+6})$ to z^2 and so it acts as a similarity: If $\mathbf{u}, \mathbf{v} \in \mathbb{F}_5^6$ then we require that

$$\begin{aligned} \langle \mathbf{u}C_5, \mathbf{v}C_5 \rangle &= \omega^2 \langle \mathbf{u}, \mathbf{v} \rangle \\ \mathbf{u}C_5 S^{-1} C_5^\top \mathbf{v}^\top &= \omega^2 \mathbf{u} S^{-1} \mathbf{v} \\ \Rightarrow \omega^{-2} C &= S^{-1} C^{-\top} S \end{aligned} \qquad \begin{aligned} \langle \mathbf{u}D_5, \mathbf{v}D_5 \rangle &= \langle \mathbf{u}, \mathbf{v} \rangle \\ \mathbf{u}D_5 S^{-1} D_5^\top \mathbf{v}^\top &= \mathbf{u} S^{-1} \mathbf{v} \\ \Rightarrow D &= S^{-1} D^{-\top} S \end{aligned}$$

We use the Standard Basis algorithm to find such a matrix S and then paste together matrices

$$\hat{C} = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & C_5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \hat{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & D_5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ -S^{-1}\mathbf{v}^\top & I_6 & 0 \\ 0 & \mathbf{v} & 1 \end{pmatrix}$$

where $\mathbf{v} \in \mathbb{F}_5^6 \setminus \{\mathbf{0}\}$. The required group is $\langle \hat{C}, \hat{D}, E \rangle$. It is clear that this group has four conjugacy classes of involutions: The class containing the central involution t_0, two classes containing elements t_1 and t_2 say that lie above class $2A$ of J_2, and another class of outer involutions. Clearly only elements in the first three of these classes centralise g. Now, $t_i g$ has order 10 and $10C$ is the only class of $\mathbb{M}$ containing elements h with $h^5 \in 2A$ and $h^2 \in 5B$. We deduce that $t_0 g \notin 10C$ since $|C_{\mathbb{M}}(10C)|/|C_{C_{\mathbb{M}}(5B)}(t_0)| \notin \mathbb{N}$. This leaves t_1 and t_2, but we find that t_1 is not in a class of 6-transpositions. Hence t_2 is a representative for the unique class of involutions in $N(5B)$ that fuse to class $2A$ of $\mathbb{M}$.

We used our program `NetFinder3.mag` with this representation and produced 135 flags that reduce to 28 nets. These 28 nets lie in 23 possible isomorphism classes. The following results are in the order the nets were discovered. We have only attempted an identification where the net group is relatively easy to identify.

Index	V	E	F	Order	Mod.	$\|G\|$	Group	Monstraliser	Quotient of Γ	NNN
5b:1:[1]	1	1	1	2	1	2	2	$2{}^{\cdot}B_2$	$\{1\}$	2a,1
5b:2:[2,4]	4	6	4	2	2	10	D_{10}	HN	A_5	2a,8
5b:3:[3]	1	2	2	2	1	4	2^2	$2^{2\cdot 2}E_6(2)$	S_3	2a,2
5b:4:[5]	2	3	3	2	2	8	D_8		$2^2{:}S_3$	
5b:5:[6]	2	2	2	2	2	6	S_3		$2^2{:}3$	
5b:6:[12,14]	20	30	12	10	2	250	$5^{1+2}{:}2$		A_5	
5b:7:[13]	1	2	2	10	5	20	D_{20}		S_3	
5b:8:[15]	2	3	3	10	10	200			$2^2{:}S_3$	
5b:9:[16,52,63]	2	2	2	10	10	150			$2^2{:}3$	
5b:10:[22]	2	3	3	2	1	8	D_8		S_3	
5b:11:[24]	2	3	3	4	2	16			S_3	
5b:12:[26]	4	5	3	10	5	120	$2 \times A_5$		S_{10}	
5b:13:[27]	6	9	5	6	3	120	$2 \times A_5$		$2^8{:}(2 \times A_9)$	
5b:14:[29]	3	5	3	6	3	48			$3^2{:}S_4$	
5b:15:[30]	2	2	2	4	4	24			$2^2{:}3$	
5b:16:[53]	8	12	6	12	4	2160	$3{}^{\cdot}S_6$	$L_2(8){:}3$	$2.2^{11}{:}S_{12}$	6f,5
5b:17:[54]	5	8	4	30	5	2160	$3{}^{\cdot}S_6$	$L_2(8){:}3$	S_{15}	6f,5
5b:18:[68]	8	12	6	4	1	16			$2^2{:}S_3$	
5b:19:[71,72]	8	12	6	4	4	400			$2^2{:}S_3$	
5b:20:[76]	8	12	6	20	4	50000			$2^2{:}S_3$	
5b:21:[89]	3	4	2	14	7	336			S_7	
5b:22:[93]	2	2	2	8	8	96			$2^2{:}3$	
5b:23:[117]	4	6	4	6	2	54			$2^2{:}3$	

(6.6) Nets Centralised By An Element Of Class 5A

If $g \in 5A$ then $N_{\mathbb{M}}(\langle g \rangle) = (5{:}2 \times \mathrm{HN}){\cdot}2$, so the centraliser is HN and we can work up to conjugacy in $G = \mathrm{HN}{:}2$.

There are only a handful of representations of G that are feasible to work with: Permutations on 1140000 or 1539000 points, 246-dimensional matrices over $\mathbb{F}_2$, or 133-dimensional matrices over $\mathbb{F}_5$. All of these representations are too difficult for `NetFinder3` to cope with, and any other representation is either on more points or in higher dimension. Therefore we must change strategy.

Specifically, the part of `NetFinder3` which tests triples for conjugacy using the `IsConjugate` function is far too slow.

We find an involution and its centraliser ourselves, rather than having Magma do it. We also produce generators for the coset action of G on the involution centraliser. Hence we can find representatives for all conjugacy classes of triples of involutions in G. Testing conjugacy of triples takes too long, so instead we "fingerprint" the triples, and so determine most of the conjugacy classes of nets in G.

(6.6.1) Finding The Conjugacy Classes Of Triples Of Involutions

The group G has just one conjugacy class of involutions $[t]$ that fuse to $\mathbb{M}$-class $2A$, and $H = C_{\mathrm{HN}:2}(t)$ is a group of shape $4{\cdot}\mathrm{HS}{:}2$.

From [52] we obtain 133-dimensional matrices, g_1 and g_2 say, with coefficients in $\mathbb{F}_5$ and that generate G. Using words supplied in [52] we obtain matrices, h_1 and h_2 say, that generate H. We use the MeatAxe to show that the module breaks up as $133 = 1 + 21 + 55 + 56$ when restricted to H. Since H is a maximal subgroup of G it must be exactly the stabiliser of the 1-dimensional subspace, $\langle \mathbf{v} \rangle$ say. Thus the images of $\langle \mathbf{v} \rangle$ under the action of G are in bijective correspondence with the cosets of H which in turn are in bijective correspondence with the $2A$-involutions in G. Thus we construct permutations on 1539000 points by permuting the 1-dimensional subspaces, and this is the permutation representation of G on its class of $2A$-involutions.

We use an SLP w in 5 generators to specify en element $w(g_1, g_2, h_1, h_2, t) \in G$, or in P by using the corresponding generators. By using redundant generators we keep the SLP length down and so reduce computing time.

As with `NetFinder3` we work with the permutation representation, this time on 1539000 points, to determine the conjugacy classes of triples of involutions. However, we now store the triple as three SLPs, (w_1, w_2, w_3), in the 5 variables g_1, g_2, h_1, h_2, t. The corresponding triple of involutions is

$$(t, t^{w_2(g_1,g_2,h_1,h_2,t)}, t^{w_3(g_1,g_2,h_1,h_2,t)})$$

Again, this keeps the SLP length down, and the SLPs are easy to find.

The associated computer programs are on the CD at `/nets/5A/` and are described in Appendix A.5.

(6.6.2) Finding The Conjugacy Classes Of Nets

It is too difficult to test conjugacy of triples of involutions in any of the three representations of G we have available. Instead we assign a "fingerprint" to each triple so that if two triples have different fingerprints then they cannot be conjugate (*cf.* Section 2.1.1 and page 59).

We generate and store a list (w_i) of random SLPs in 3 variables and for a triple (a, b, c) produce a list $(\mathrm{rk}(w_i(a, b, c) - w_j(a, b, c)))$ which is the fingerprint. We also store a corresponding list $((i, j))$ that indicates how to calculate the fingerprint. The `netRF` gains two new variables to store these lists, `n` and `ni` respectively.

In `NetFinder63` we have a list, `nets`, of flags. Each of these flags is stored as a `netRF` variable. We produce the fingerprints as follows:

1. Initialise by setting `nets[i].n := [0]` and `nets[i].ni[1] := [0,0]`.
2. For each set X of triples with the same invariants:
   (a) Let k be the highest index of SLP used in the invariants of the triples in X.
   (b) Calculate $r_{k,j}(x) = \mathrm{rk}(w_k(a, b, c) - w_j(a, b, c))$ for $1 \leqslant j \leqslant k$ and all representatives (a, b, c) of elements $x \in X$.
   (c) If $\{r_{k,j}(x) \mid x \in X\}$ contains distinct values then append $r_{k,j}(x)$ to `x.n` and append (k, j) to `x.ni`. Otherwise increment k and repeat until either a distinguishing invariant is found, or the SLPs are exhausted.

We find that using just 24 of the SLPs we can obtain 10267 unique fingerprints for the 10376 conjugacy classes of triple. The calculations were split up according to the net order, and the resulting nets of order n are stored in `/nets/5A/HN2nets64/nets`n`.mag`. We obtain at most 422 distinct conjugacy classes of nets:

Net order	Number of flags	Number of nets
1	1	1
2	77 (75 invariants)	21
3	55 (46 invariants)	8
4	353 (326 invariants)	41
5	242 (227 invariants)	19
6	672	52
7	350 (294 invariants)	15
8	804	30
9	810	24
10	911	48
11	869	22
12	1029	35
14	602	20
15	831	22
19	513	10
20	913	24
21	168	4
22	209	6
25	475	8
30	174	5
35	238	5
40	80	3

(6.7) Further Work

Clearly the work on nets is not finished. There are a number of areas which require further investigation.

(6.7.1) Nets Centralised By An Element Of Class 3B

The normaliser of a group generated by an element of class $3B$ is $3^{1+12} \cdot 2 \cdot \mathrm{Suz}{:}2$. It has just one conjugacy class, C say, of involutions that fuse to class $2A$ of $\mathbb{M}$. Finding the conjugacy classes of triples of involutions in $6 \cdot \mathrm{Suz}{:}2$ is easy, but there are many more involutions which are not in this complement.

In the notation of Chapter 5 there is a unique class of involutions in $G = 3^{1+12}{:}6 \cdot \mathrm{Suz}{:}2$ which projects to C. The label of this class is $(4, 2, 0)$. Two other classes of G project onto class C, but these other classes contain elements of order 6.

Let a be in class $-2A$ of $6 \cdot \mathrm{Suz}{:}2$ and let $X \subset 3^{12}$ be such that $(a, \mathbf{v}, \lambda) \in G$ is in class $(4, 2, 0)$ for all $\mathbf{v} \in X$. For given a and $\mathbf{v}$ there is only one possibility for λ because class $(4, 2, 1)$ (conjugate to $(4, 2, 2)$) is distinct from class $(4, 2, 0)$. It is easy to find λ because the two incorrect values give an element of order 6. Thus the elements of C are in bijective correspondence with the elements $(a^g, \mathbf{v}g, \lambda)$, where λ is determined by a and $\mathbf{v}$. Henceforth we shall ignore λ, and essentially work in $H = 3^{12}{:}6 \cdot \mathrm{Suz}{:}2$.

We can take the first involution as $(a, \mathbf{0})$ say. Then $C_H(a, \mathbf{0}) = (\operatorname{Fix} a){:}C_{6 \cdot \mathrm{Suz}:2}(a)$ where $\operatorname{Fix} a$ is the fixed space of a on 3^{12}. The second involution of a triple is one of the orbit representatives $(b_i, \mathbf{v}_{i_j})$ for the action of $C_H(a, \mathbf{0})$ on class $(4, 2, 0)$. Clearly the b_i are orbit representatives for the action of $C_{6 \cdot \mathrm{Suz}:2}(a)$ on the class $[a]_{6 \cdot \mathrm{Suz}:2}$. If $a^g = b$ then the $\mathbf{v}_{i_j}$ lie in $Y = \{\mathbf{v}g \mid \mathbf{v} \in X\}$ which will break into a number of suborbits. In particular, we must find the orbits on Y of $(\operatorname{Fix} a){:}C_{6 \cdot \mathrm{Suz}:2}(a, b)$.

The centraliser in H of $(a, \mathbf{0})$ and $(b, \mathbf{v})$ is $(\operatorname{Fix} a \cap \operatorname{Fix} b){:}(C_{6 \cdot \mathrm{Suz}:2}(a, b) \cap \operatorname{stab} \mathbf{v})$. Therefore for each choice of $(b, \mathbf{v})$ we expect to obtain different possibilities for the third involution $c \in 6 \cdot \mathrm{Suz}{:}2$. To find the $(c, \mathbf{w})$ we must act on $Z = \{\mathbf{v}g | \mathbf{v} \in X\}$ where $a^g = c$ by $(\operatorname{Fix} a \cap \operatorname{Fix} b){:}(C_{6 \cdot \mathrm{Suz}:2}(a, b, c) \cap \operatorname{stab} \mathbf{v})$.

Having found the various representatives we can paste them into 38-dimensional matrices using the correct value of λ. We can then find words for these in the standard generators.

There may be some advantages in continuing to use the 38-dimensional representation when assembling the triples into nets. Given a triple of involutions $((g_i, \mathbf{v}_i, \lambda_i) \mid 1 \leqslant i \leqslant 3)$ we must determine which of our standard triples it is conjugate to:

(i) Find $\mathbf{w}_1 \in 3^{12}$ that moves $\mathbf{v}_1$ to $\mathbf{0}$.

(ii) Find $h_1 \in 6 \cdot \mathrm{Suz}{:}2$ that conjugates g_1 to a.

(iii) Find $h_2 \in C_{6 \cdot \mathrm{Suz}:2}(a)$ that conjugates $g_2^{h_1}$ to a standard orbit representative b.

(iv) Find $(k_2, \mathbf{w}_2) \in (\operatorname{Fix} a){:}C_{6\cdot\mathrm{Suz}:2}(a, b)$ that moves $\mathbf{v}_2 h_1 h_2$ to a standard orbit representative $\mathbf{v} \in Y$.

(v) Use an element of $C_{6\cdot\mathrm{Suz}:2}(a, b) \cap \operatorname{stab} \mathbf{v}$ to conjugate $g_3^{h_1 h_2 k_2}$ to a standard representative c.

(vi) Use an element of $(\operatorname{Fix} a \cap \operatorname{Fix} b){:}(C_{6\cdot\mathrm{Suz}:2}(a, b, c) \cap \operatorname{stab} \mathbf{v})$ to conjugate $\mathbf{v}_3 h_1 h_2 k_2 h_3$ to a standard representative $\mathbf{w} \in Z$.

(6.7.2) Nets Centralised By An Element Of Class 3A

The normaliser of an element of class $3A$ is $3{\cdot}\mathrm{Fi}_{24}$ which has representations on just under a million points, or as 1566-dimensional matrices over $\mathbb{F}_2$. Furthermore, there are 85119 conjugacy classes of triples of involution, so this is well beyond the scope of our existing programs.

The cases $N(2A) = 2{\cdot}B$ and $N(2B) = 2^{1+24}{\cdot}\mathrm{Co}_1$ are similarly challenging.

Re-considering the case of $5A$, we had a correspondence between involutions, cosets of the involution centraliser (obviously), and 1-dimensional subspaces of $\mathbb{F}_5^{133}$. Let t be a chosen involution and $\mathbf{v}_t$ be the corresponding vector (we may choose $\mathbf{v}_t \neq \mathbf{0}$ at will in the fixed 1-space). The vector corresponding to t^g is then $\mathbf{v}_t g$, so we can switch between involutions and vectors.

The large orbit enumeration algorithms of Lübeck and Neunhöffer [32] are particularly suited to the case of matrices acting on vectors, and scale well to the sort of very large orbit we will wish to consider. Furthermore, if $\mathbf{v}$ is the seed vector in the orbit being enumerated, then for any $\mathbf{w}$ in the orbit the algorithms provide an element g such that $\mathbf{v}g = \mathbf{w}$, so the ambition of switching between vectors and involutions is realistic.

This method may also be applicable for the $N(2A)$ case where the 4370-dimensional $\mathbb{F}_2$-module restricted to $2{\cdot}^2\mathrm{E}_6(2){:}2$ contains four 1-dimensional modules in its composition series.

(6.7.3) Nets That Generate The Monster

If the centraliser of $\langle a, b, c\rangle$ is trivial, then we have no option but to work in $\mathbb{M}$. However, each $2A$ involution a has a corresponding vector $\mathbf{t}_a$ in the 196884-dimensional Griess Algebra (see pp. 230 of [8]), so again we can use the orbit algorithms to find the triples of transposition. Even more useful is algebra product that is invariant under the action of $\mathbb{M}$. The action of a by conjugation on the class $2A$ of involutions can be recovered from $\mathbf{t}_a$ by using this product, so we can apply the braiding operations and thus assemble the nets just by using the vectors, and not 196884-dimensional matrices.

Appendix A

The CD

In some cases files have been compressed in order to save space. In this case `file.abc` described below will be replaced on the CD with `file.abc.gz`. Similarly, an entire folder `dir` may be replaced with `dir.tar.gz`.

(A.1) A Computer Construction Of The Monster

`/monster/mop2` The 196882-dimensional $\mathbb{F}_2$ representation of $\mathbb{M}$ and driver programs.

`*.m`: Data files. Contain the generators B, C, E, and T.

`char.c`: Driver program. Calculate the character value of a word. See section 2.3.3.

`cid_data_c.txt`: Data file. Conjugacy class identification data, as in Appendix B formatted for easy use from C driver programs.

`class2.c`: Driver program. Calculate mod 2 character values required for class identification. See section 2.3.3.

`mel2436.c`: Driver program. Calculates element orders on the 142-dimensional module for $2^{\cdot}$Suz:2. If this divides 24 or 36 then it calculates the correct order and logs the results.

`mels.c`: Driver program. Like `melts.c` but for words with no T. See section 2.3.3.

`melts.c`: Driver program. Calculates element orders for words of the form WT for $W \in \langle B, C, E\rangle$. See section 2.3.3.

`mop.h`: Parker & Wilson's implementation of the mod 2 construction of $\mathbb{M}$.

`mor2.h`: Driver functions. Useful functions, mostly for working with `suzel`s, that are not part of `mop2.h`.

`order.vec?`: Data file. The two vectors used for calculating element orders exactly.

`suzchar.c`: Driver program. As `char.c` but for elements of $\langle B, C, E \rangle$.

`v?.vec`: Data files.

`vorder.vec`: Data file. Vector used for guessing element orders.

`/monster/mop7` The 196883-dimensional $\mathbb{F}_7$ representation of $\mathbb{M}$ and driver programs.

`*.m7`: Data files. Contain the generators A, C, D, E, and T.

`char7.c`: Driver program. Calculate the character value of a word. See section 2.3.3.

`cid_data_c.txt`: Data file. Conjugacy class identification data, as in Appendix B formatted for easy use from C driver programs.

`class7.c`: Driver program. Calculate mod 7 character values required for class identification. See section 2.3.3.

`factor.c`: Produces a generator for the 3-group that must be factored out of 3^{1+12}:6·Suz:2 to obtain the maximal subgroup $N(3B)$ of $\mathbb{M}$.

`make+8b.c`: Produces elements in 6·Suz:2 lying above class 8B of Suz, as described in Section 3.3.1.

`mop7.h`: Wilson's implementation of the mod 7 construction of $\mathbb{M}$, modified to include grease as described in Section 2.3.2.

`mor7.h`: Driver functions. Useful functions, some translated from `mop2.h` and `mor2.h`.

`suzchar.c`: Driver program. Calculate character values. See section 2.3.3.

`suzchar24.c`: Driver program. Calculates character values of elements of 6·Suz:2.

`vorder.v7`: Data file. Vector for guessing element orders.

(A.2) Conjugacy Class Representatives In The Monster Group

`/monster/elts`

Monster elements database for words of the form WT where $W \in \langle B, C \rangle$.

`data/`: Data directory. Contains one file `n.data` for each element order n in the Monster. The data file format is

`WT|n|i|j|k|l|m|n|p|r|`

where:

- i is the trace of the element modulo 2.
- j is the trace of the element modulo 2 after the first power map has been applied.
- k is the trace of the element modulo 2 after the second power map has been applied.
- l is the trace of the element modulo 2 after the third power map has been applied.
- m is the trace of the element modulo 7.
- $\vdots$

Very few of the traces have been computed, so most records contain only the word and element order.

`OtoData.pl`: Take output from order calculating program (`/monster/mop2/melts.c`) and add words to database.

`heapdata.pl`: Heapsort the data files. This makes finding words easy.

`cid.data`: Conjugacy class identification data, as shown in Appendix B.

`pmap.data`: Power map data encoded for the Perl scripts. Format is

`n|1|p|q|r`

where n is the element order, and p, q, and r are the powers as used in the ATLAS.

`MtoData.pl`: Add output from trace calculating programs (`/monster/mop2/char.c` and `/monster/mop7/ch` to the data files.

`monster.pm`: Main element finding function.

`find.pl`: Find monster elements.

`cid.pl`: Print out an element in each conjugacy class.

`numbers.pl`: Print out the number of elements found in each conjugacy class.

`/monster/Suzuki`

Monster elements database for words of the form W where $W \in \langle B, C, E\rangle$. Files are adapted from `/monster/elts`. In addition:

`BC guess/`: Estimate orders of words in $\langle B, C\rangle$ by using the projection onto the 142-dimensional $\mathbb{F}_2$-representation of $2{}^{\cdot}\mathrm{Suz}$.

`142.gap` contains generators for this representation.

`gapify.pl` produces a GAP program from a list of words.

`job.pl` splits the GAP program into smaller jobs.

`select.pl` selects words from the output that have interesting element orders.

The remaining files are some input and output.

`E/`: Word generation programs and lists for words $W \in \langle B, C, E\rangle$.

(A.3) Fischer Matrices

`/Fischer`

`cl4.gap`: Correct construction of the character table of $2^2{}^{\cdot}\mathrm{Fi}_{22}{:}2$. This should be treated as a draft version of `ct22g2.gap`.

`ct22g2.gap`: Function to construct the character table of a group of shape $(2 \times 2{}^{\cdot}G){:}2$, and some example input.

`pres.mag`: Presentations for the isoclinic groups $(4 \circ 2{}^{\cdot}\mathrm{A}_6){}^{\cdot}2_3$ and $(2 \times 2{}^{\cdot}\mathrm{A}_6){}^{\cdot}2_3$.

(A.4) The Character Tables Of A Maximal Subgroup Of The Monster, And Some Related Groups

`/character`

`classes/mod5/`: Characteristic 5 representations used in Section 5.1. See `README.txt` for explanations of the file names.

`classes/char0.mag`: Characteristic zero representation of $6{}^{\cdot}\mathrm{Suz}$ [5].

`12+12/`: The script `12.12.sh` takes the 12-dimensional $\mathbb{F}_3$ representation of $\cdot 0 \cong \mathrm{Co}_0 \cong 2{}^{\cdot}\mathrm{C0}_1$

and uses words from the web Atlas to find a subgroup 6·Suz:2. The resulting representation of 6·Suz:2 is the same as that given on the web Atlas.

The files `C.1` and `D.1` are standard generators C and D for 6·Suz:2 in MeatAxe binary format.

`13/`: Construction of a 13 dimensional module for 3^{12}:2·Suz:2.

`6s2/`: Calculation of the conjugacy classes of 3^{12}:6·Suz:2.
`repW.txt` contains SLPs for the class representatives for 6·Suz:2 calculated by Magma. `cl1.mag` performs the coset analysis described in Section 5.1.2 and `cl2file.mag` prints the results to `liftings.mag`.

`mags/`: Contains the representations of 2·Suz:2 and 6·Suz:2 and a matrix for the invariant symplectic form. These files are used by many of the other Magma programs.

`1+12+1`: In `bilinearForm.sh` we use the MeatAxe to find the symplectic form stabilised by 2·Suz. In `BCE.sh` we find the third standard generator E and paste together the 14-dimensional representation of 3^{1+12}:2·Suz:2 as described in Section 5.1.2.

`38/`: Paste together the 12-dimensional representation in `12/` and the 14-dimensional representation in `1+12+1/` to make the 38-dimensional representation described in Section 5.1.2. The script `makewonky.sh` makes the version with re-ordered co-ordinates.

`V6s2/`: The calculations described in Section 5.1.3 are performed by the program `pasting39.case.mag`. The program `pasting37.mag` is an earlier version.

`makewords.mag` outputs SLPs for the conjugacy class representatives of 3^{1+12}:6·Suz:2 and their labels (i, j, k).

The function in `pasting37functions.mag` which constructs a 38-dimensional matrix from an SLP, a vector, and an element of $\mathbb{F}_3$ produces the matrices we use to find the power maps and quotient map as described in Section 5.2.

`729/`: Construction of 1458-dimensional module for 3^{1+12}:2·Suz:2 over $\mathbb{F}_{103}$. Described in Section 5.3.

`729.31`: Construction of 1458-dimensional module for 3^{1+12}:2·Suz:2 over $\mathbb{F}_{31}$. Described in Section 5.3.5.

`vals/`: Contains Magma files for each of the three constructions, $\mathbb{F}_{103}$, $\mathbb{F}_{223}$, and $\mathbb{F}_{31}$.

`check.mag` calculates element orders. We used this to determine whether we need to replace C with C^{-1}.

`traces729.mag` calculates the traces modulo 22969 by using the $\mathbb{F}_{103}$ and $\mathbb{F}_{223}$ representations.

`tracesEV.mag` uses the $\mathbb{F}_{31}$ representation to specify the trace as a certain power of a root of a Conway polynomial. This is used to lift the traces to $\mathbb{C}$.

`induction/`: Programs for inducing characters as described in Section 5.5.

`pf.mag` The permutation representation of 3^{12}:6·Suz:2 on $98280 + 5346$ points.

`gens.mag` locates the generators e', c_2, c_3, u_1, u_2 in the 103626 point permutation representation. The generator u_2 is stored in the file `zzz.mag` in order to speed things up.

`make.mag` is the basic induction program as described at the start of Section 5.5.5.

`makeQ.mag` is the induction program modified to include the improvements described at the end of Section 5.5.5. It assumes the module M is irreducible.

`makeQsMods.mag` prepares the two constituents of M in the case when it is reducible. These are written out to files which, after editing, are used with `makeQs.mag`.

`makeQs.mag` is `makeQ.mag` for use with the modules output from `makeQsMods.mag`.

`makeQno3.mag` is for the case $M_1 \otimes M_3$.

`induction/modules`: A selection of modules for use with the induction programs, and the programs we used to make them. `15.mod1s` was produced with `makeQsMods.mag`.The file n`.mag` contains the module corresponding to GAP character `X.`n of $U_5(2)$:2. Files with names such as `gg120a.mag` were converted from the C MeatAxe.

`monster/fusion 6s2 to M`: Use Chapter 3 to find the conjugacy class in $\mathbb{M}$ of the projection to 3^{1+12}·2·Suz:2 for each class representative of 3^{1+12}:6·Suz:2. The results are in `monster/results_a`

The contents of this folder convert the Magma SLPs for use with the `mop` and `mop7` programs, then re-format the results.

`monster/get65520.out`: Output from `mop` program to extract trace of 65520-dimensional module for 3^{12}:2·Suz:2.

`monster/ordersQuo.txt`: Orders of class representatives of 3^{1+12}:6·Suz:2 when projected down to 3^{1+12}:2·Suz:2. Because the 1458-dimensional module is for 3^{1+12}:2·Suz:2 the degrees of the Conway polynomials we used in Magma correspond to the element orders in this group.

`monster/tablecover.gap`: Constructs the (partial) character table. Also contains the results of the power map calculations.

`monster/induced.gap`: Reads in the results in `monster/inductions/` of the induction calculations, assembles the characters, and does some basic checks.

(A.5) Nets

For $n \in \{31, 23, 19, 13, 17, 11, 7, 5B\}$ each of the folders `/mets/n/` contains a Magma file that initialises a group `G` and an involution `clG[1]`. After loading one of these files loading `/nets/NetFinder3.mag` will find all the nets up to conjugacy in `G`, as described in Section 6.3.2 and Section 6.3.3. After running `NetFinder3.mag` we saved the Magma workspace, we also output the nets to a text file, `/nets/n/results.mag`.

For nets centralised by an element of class $5A$ there is a sequence of procedural programs to be run. The nets are stored in files, one file for each order of net. We run:

`/nets/NetFinder51.mag`: Finds the conjugacy classes of triples of involutions. We store a net for each of these, and write them out to disc.

`/nets/5A/NetFinder62.5.mag`: Prepares the group $G \cong$ HN:2, an involution, and its centraliser $H \cong 4{\cdot}$HS:2. Loads one of the nets files from `NetFinder51.mag`.

`/nets/NetFinder63.mag`: Makes the fingerprints. See Section 6.6.2.

`/nets/5A/NetFinder631.mag`: For nets with the same fingerprints make the nets and determine whether they could be conjugate.

`/nets/NetFinder64.mag`: For the remaining conjugacy classes of triples, make the nets and reduce the list to conjugacy classes of nets.

Appendix B

Conjugacy Class Identification

The table below gives a method for determining the conjugacy class in $\mathbb{M}$ of an element of known order. The format is easily read by machine (thus is a little cryptic). For each order n of element in $\mathbb{M}$ there is a header row followed by one line for each class of elements of order n. The header row shows what needs to be calculated: "`2`" for a trace modulo 2, "`7`" for a trace modulo 7, and "`aPb`" for the trace modulo `a` of the `b`th power. The subsequent rows give the traces for the class indicated in the first column. A header row containing an asterisk indicates that classes of order n cannot be distinguished. A header row containing the number `1` indicates that there is only one conjugacy class of elements of order n, so no traces need to be calculated.

The class names are as in the ATLAS [8] with the exception that "$40CD$" means "$40C$ or $40D$", for example. The following example shows how the table can be used. Let g have order 12, then:

- If the character value is $0 \mod 7$ then $g \in 12A$.
- If the character value is $5 \mod 7$ then calculate the character value mod 2.
  - If this is 1 then the $g \in 12E$.
  - If this is 0 then calculate the character value of g^2, modulo 7. If this is 0 then $g \in 12F$, and if 5 then $g \in 12B$.
- Similar for if the character value mod 7 is 6.

For more information see Section 3.1.

```
2   7
B   2
A   3

3   7
B   4
A   5
C   6

4   7   7P2
D   1
A   2   2
B   2   3
C   5

5   2
A   0
B   1

6   7   2
B   0   0
C   0   1
A   1
D   4
E   5
F   6

7   2
B   0
A   1

8   7   7P2
A   0
D   2
E   3
B   4
F   6   1
C   6   2

9   2
A   1
B   0

10  7   2
A   0   0
E   0   1
C   3
B   5
D   6

11  2
A   1

12  7   2   7P2 7P3
A   0
I   1
G   4
F   5   0   0
B   5   0   5
E   5   1
J   6   0       1
D   6   0       2
H   6   0       5
C   6   1

13  2
A   0
B   1

14  2   7
C   0
B   1   2
A   1   3

15  2   2P3
A   0   0
D   0   1
B   1   0
C   1   1

16  7
A   3
B   6
C   1

17  1

18  2   2P3 7
A   0       4
E   0       5
D   0       6
B   1   0
C   1   1

19  1

20  7   2
A   5
B   1
C   0
D   4   0
E   2
F   4   1

21  2   2P7 7
C   0   0
A   0   1
D   1       4
B   1       5

22  7
B   0
A   4

23  *

24  7   7P2 7P3
H   1
A   2   0
B   2   5
I   3
D   5
C   6   1
J   6   6   6
G   6   4
F   6   5
E   6   6   4

25  1

26  2
A   0
B   1

27  *

28  2   7   7P2
D   0
B   1   2   2
A   1   2   3
C   1   5

29  1

30  2   2P3 7   7P3
B   0   0   2
C   0   0   6
E   0   1
D   1   0
G   1   1       0
F   1   1       3
A   1   1       6

31  *

32  7P2
```

```
B    1
A    6

33   2
B    0
A    1

34   1

35   2
B    0
A    1

36   7
C    0
D    1
A    2
B    6

38   1

39   2    2P3
A    1
B    0    0
CD   0    1

40   2    7
A         6
B         1
CD   1

41   1

42   2    2P7
C    0    0
A    0    1
B    1    0
D    1    1

44   *
```

```
45   1

46   7
CD   1
AB   6

47   *

48   1

50   1

51   1

52   2
A    0
B    1

54   1

55   1

56   2
BC   0
A    1

57   1

59   *

60   2    2P3  7
A    0    0    1
B    0    0    6
F    0    1
E    1    0
C    1    1    0
D    1    1    5

62   *

66   2
```

```
A    0
B    1

68   1

69   *

70   2
B    0
A    1

71   *

78   2
BC   0
A    1

84   2    2P7
C    0    0
A    0    1
B    1

87   *

88   *

92   *

93   *

94   *

95   *

104  *

105  1

110  1

119  *
```

List of References

[1] F. Ali and J. Moori, *The Fischer-Clifford matrices of a maximal subgroup of* Fi_{24}' , Representation Theory (electronic) **214** (2000), 1–56.

[2] ———, *The Fischer-Clifford matrices and character table of a maximal subgroup of* Fi_{24} , Preprint, 2003.

[3] T. Apostol, *Modular functions and dirichlet series in number theory*, Graduate Textx in Mathematics, no. 41, Springer-Verlag, 1990.

[4] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, Inventiones Mathematicae **76** (1984), 469–514.

[5] J. N. Bray, *A characteristic zero representation of* $6{}^{\cdot}\mathrm{Suz}$ *in 12 dimensions*, private communication, 2004.

[6] ———, *A characteristic zero representation of* $\mathrm{U}_5(2)$ *in 10 dimensions*, private communication, 2005.

[7] G. Butler, *Fundamental algorithms for permutation groups*, Lecture Notes In Computer Science, no. 559, Springer-Verlag, 1991.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, $\mathbb{ATLAS}$ *of finite groups*, Oxford University Press, 1985.

[9] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bulletin of the London Mathematical Society **11** (1979), 308–339.

[10] C. Dong, H. Li, and G. Mason, *Modular invariance of trace functions in orbifold theory*, Communications in Mathematical Physics **214** (2000), 1–56.

[11] B. Fischer, *Clifford matrices*, Representation Theory of Finite Groups and Finite-Dimensional Lie Algebras, 1991, pp. 1–16.

[12] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics, no. 134, Academic Press, 1988.

[13] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE – *A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras*, Appl. Algebra Engrg. Comm. Comput. **7** (1996), 175–210.

[14] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups*, Mathematical Surveys and Monographs, no. 40.1, American Mathematical Society, 1994.

[15] R. L. Griess, *The Friendly Giant*, Inventiones Mathematicae **69** (1982), 1–102.

[16] P. E. Holmes, S. A. Linton, and S. H. Murray, *Product replacement in the monster*, Journal of Experimental Mathematics **12** (2003), 123–126.

[17] P. E. Holmes and R. A. Wilson, *A new maximal subgroup of the Monster*, Journal of Algebra **251** (2001), 435–447.

[18] ———, *A computer construction of the monster using 2-local subgroups*, Journal of the London Mathematical Society **67** (2003), 349–364.

[19] ———, PSL(2, 59) *is a subgroup of the monster*, Journal of the London Mathematical Society **69** (2004), 141–152.

[20] T. Hsu, *Quilts: Central extensions, braid actions, and finite groups*, Lecture Notes in Mathematics, no. 1731, Springer-Verlag, 2000.

[21] I. M. Isaacs, *Character theory of finite groups*, Dover, 1994.

[22] G. D. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, 1981.

[23] C. Jansen, K. Lux, R. A. Parker, and R. A. Wilson, *An atlas of Brauer characters*, London Mathematical Society Monographs, New Series, no. 11, Oxford University Press, 1995.

[24] C. Jansen and J. Müller, *The 3-modular decomposition numbers of the sporadic simple Suzuki group*, Communications in Algebra **25** (1997), 2437–2458.

[25] G. Karpilovsky, *Projective representations of finite groups*, Monographs and textbooks in pure and applied mathematics, no. 94, Marcel Dekker, 1985.

[26] P. Kleidman and M. Liebeck, *The subgroup structures of the finite classical groups*, London Mathematical Society Lecture Note Series, no. 129, Cambridge University Press, 1990.

[27] Lehrstuhl D für Mathematik, RWTH Aachen, *The MeatAxe—Computing with Modular Representations*, 2004, (`http://www.math.rwth-aachen.de/homes/MTX/`).

[28] M. Liebeck, C. Praeger, and J. Saxl, *On the O'Nan-Scott theorem for finite primitive permutation groups*, Journal of the Australian Mathematical Society **44** (1988), 389–396.

[29] S. A. Linton, R. A. Parker, P. Walsh, and R. A. Wilson, *Computer construction of the monster*, Journal of Group Theory **1** (1998), 307–337.

[30] F. Lübeck, *Conway polynomials for finite fields*, 2004, (`www.math.rwth-aachen.de:8001/~Frank.Luebeck/data/ConwayPol`).

[31] F. Lübeck et al., *The modular atlas*, 2004, (`www.math.rwth-aachen.de/~MOC`).

[32] F. Lübeck and M. Neunhöffer, *Enumerating large orbits and direct condensation*, Experimental Mathematics **10** (2001), 197–205.

[33] U. Meierfrankenfeld and S. Shpectorov, *The maximal 2-local subgroups of the Monster and the Baby Monster*, Preprint, 2002.

[34] J. Moori, *On the groups G^+ and $\overline{G}$ of the forms $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$*, Ph.D. thesis, University of Birmingham, 1975.

[35] J. Moori and Z. Mpono, *The Fischer-Clifford matrices of the group $2^6{:}Sp_6(2)$*, Quaestiones Mathematicae **22** (1999), 257–298.

[36] S. P. Norton, *The uniqueness of the Monster*, Finite Groups—Coming of Age (John McKay, ed.), Contemporary Mathematics, vol. 45, American Mathematical Society, 1982, pp. 271–285.

[37] ______, *Generalized moonshine*, The Arcata Conference on Representations of Finite Groups, Proceedings of Symposia in Pure Mathematics, vol. 47, 1987, pp. 208–210.

[38] ______, *Anatomy of the Monster: I*, The ATLAS of Finite Groups: Ten Years On (R. T. Curtis and R. A. Wilson, eds.), London Mathematical Society Lecture Note Series, vol. 249, 1998, pp. 198–214.

[39] ______, *Netting the Monster*, The Monster and Lie Algebras (J. Ferrar and K. Harada, eds.), Ohio State University Mathematical Research Institute Publications, vol. 7, 1998.

[40] ______, *Counting nets in the Monster*, Combinatorics and Geometry (Durham, 2001) (A. A. Ivanov, M. W. Liebeck, and J. Saxl, eds.), World Science Publishing Company, 2003, pp. 227–232.

[41] ______, *A string of nets*, personal communication, 2003.

[42] R. A. Parker, *The computer calculation of modular characters (the meat-axe)*, Computational Group Theory (M. D. Atkinson, ed.), Academic Press, 1984, pp. 267–274.

[43] R. A. Parker and R. A. Wilson, *The computer construction of matrix representations of finite groups over finite fields*, Journal Of Symbolic Computation (1990), 583–590.

[44] R. B. Salleh, *On the constructions of the character tables of group extensions*, Ph.D. thesis, The University of Birmingham, 1982.

[45] I. A. I. Suleiman, P. G. Walsh, and R. A. Wilson, *Conjugacy classes in sporadic simple groups*, Communications in Algebra **28** (2000), 3209–3222.

[46] M. P. Thorne, *On Fischer's "Monster"*, Ph.D. thesis, University of Birmingham, 1978.

[47] University of Sydney, *The Magma Handbook*, 2004, (`http://magma.maths.usyd.edu.au/magma/htmlhelp/MAGMA.htm`).

[48] P. G. Walsh, *Computational study of the Monster and other sporadic simple groups*, Ph.D. thesis, University Of Birmingham, 1996.

[49] R. A. Wilson, *The complex Leech lattice and maximal subgroups of the Suzuki group*, Journal of Algebra **84** (1983), 151–188.

[50] ______, *A construction of the monster group over $GF(7)$ and an application*, Preprint, 2000.

[51] ______, *Conjugacy class representatives in Fischer's baby monster*, London Mathematical Society Journal of Computation and Mathematics **5** (2002), 175–180.

[52] R. A. Wilson et al., *A world wide web atlas of group representations*, 1995–2005, (`brauer.maths.qmul.ac.uk/Atlas`).