MTH6109 Combinatorics

Assignment 9 For handing in on 8 December 2011

As usual, hand in a selection of questions of your choice to the red box on the ground floor by 3.30pm on Thursday 8th December.

The topic of this sheet is Latin squares.

1 A Latin square of order n is said to be in *standard form* if the entries along its first row and down its first column are $1, 2, 3, \ldots, n$ in that order.

(a) Show that there is exactly one Latin square in standard form of order 3.

(b) Find four different Latin squares in standard form of order 4.

2 Use the field of integers modulo 5 to construct a set of four MOLS of order 5.

3 Show that, up to permutations of rows and columns and changes in the names of the symbols, there are just two different Latin squares of order 4. Show that one, but not the other, has an *orthogonal mate* (a Latin square orthogonal to it).

4 Prove that the Latin square given by the addition table of the integers mod n has an orthogonal mate if and only if n is odd.

5 We proved the following result in Exercises 8, Q3.

If $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ is a family of sets which satisfy Hall's condition, then some set $A_i \in \mathcal{F}$ has the property that, for EVERY element $a_i \in A_i$, there is an SDR for $\mathcal{F}$ which uses a_i as a representative for A_i.

Use this result to give an alternative proof that if $A_1, A_2, \ldots, A_n$ is are sets which satisfy Hall's condition, and $|A_i| \geq r$ for all $i = 1, 2, \ldots, n$, then it has at least $f(r, n)$ different SDRs where

$$f(r,n) = \begin{cases} r! & \text{if } r \leq n \\ r(r-1)\ldots(r-n+1) & \text{if } r \geq n \end{cases}$$

Hint: this is an 'easy' induction on n.

6 Let A and B be orthogonal Latin squares of order n, which use the symbols $0, 1, \ldots, n-1$. Construct a matrix S in which the (i,j)th entry consists of the pair (a_{ij}, b_{ij}), regarded as a 2-digit number written in base n. Show that S has the properties

(a) its entries are all the integers from 0 to $n^2 - 1$ inclusive;

(b) the sum of the entries in any row or column is $n(n^2 - 1)$.

(This construction is due to Euler.)