Queen Mary University of London

MTH6109 Combinatorics

Assignment 8 — For handing in on 1 December 2011

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1 Determine whether each of the following collections of sets have a SDR. In each case, justify your answer by either giving a SDR or explaining why there is no SDR.

(a) The sets $A_1, A_2, A_3, A_4, A_5, A_6$, where $A_1 = \{1, 3, 5\}$, $A_2 = \{2, 3, 4, 6\}$, $A_3 = \{1, 5\}$, $A_4 = \{2, 3, 6\}$, $A_5 = \{1, 3\}$, and $A_6 = \{3, 5\}$.

(b) The sets $B_1, B_2, B_3, B_4, B_5, B_6$, where $B_1 = \{1, 3, 5\}$, $B_2 = \{2, 3, 4, 6\}$, $B_3 = \{1, 5\}$, $B_4 = \{2, 3, 6\}$, $B_5 = \{1, 3\}$, and $B_6 = \{4, 5\}$.

2 (a) For each $i \in \{1, 2, 3, 4, 6\}$, find a collection of three subsets of $\{1, 2, 3\}$ which has exactly i different SDRs.

(b) Can this be done for $i = 5$? Justify your answer.

3 Adapt the inductive proof of Hall's theorem given in lectures to prove the following stronger result:

If $A_1, A_2, \ldots, A_n$ are sets which satisfy Hall's condition, then some set A_i has the property that, for EVERY element $a_i \in A_i$, there is an SDR which uses a_i as a representative for A_i.

4 (The deficit form of Hall's theorem.)

Let $A_1, \ldots, A_n$ be subsets of a set X. Suppose that, for some positive integer m,

$$\left| \bigcup_{j \in J} A_j \right| \geq |J| - m$$

for all $J \subseteq \{1, 2, 3, \ldots, n\}$.

Prove that it is possible to find $n - m$ of the sets $A_1, \ldots, A_n$ which have a SDR. [Hint: add m 'dummy' elements $z_1, \ldots, z_m$ to all the sets A_i.]