MTH6109 **Combinatorics**

Assignment 4 **For handing in on 27 October 2011**

Your solution to a selection of questions of your own choice should be handed in to the RED box on the GROUND floor by 3.30pm on Thursday.

Include with your solution a brief *self-assessment*, explaining how well you have understood the week's work, any areas of particular difficulty, why you have chosen to hand in these particular question(s), how well you think you have answered it, and (important:) what help you had with it.

I will mark a reasonable amount of your solutions, and hand back marked work (and discuss it with you, where necessary) at the tutorials the following week.

The learning objectives for this week are: understanding generating functions, in particular using generating functions to solve linear recurrence relations with constant coefficients.

If these questions are too easy, then there are some more challenging questions on last year's exercises which you can find on the module web-site.

1 Let $B(n)$ be the nth Bell number, the number of partitions of $\{1,\ldots,n\}$. Prove directly that $B(n) \leq n!$ for all n.

2 Let F_n be the nth Fibonacci number. Prove by induction that

$$F_n{}^2 - F_{n-1}F_{n+1} = (-1)^n$$

for all $n \geq 1$.

3 Let (r_n) be a sequence of numbers defined by the recurrence relation $r_n = r_{n-1} + 2r_{n-2}$ for $n \geq 2$, and the initial conditions $r_0 = -1, r_1 = 4$.

(a) Use the characteristic equation of the recurrence relation to determine r_n for all $n \geq 0$.

(b) Use the recurrence relation to show that the generating function for the sequence (r_n) is $(-1+5t)/(1-t-2t^2)$.

(c) Determine the numbers (r_n) for all $n \geq 0$ by expanding the generating function using partial fractions.

4 In last week's exercises, you calculated a recurrence relation for C_n, the number of partitions of a set of size n into subsets of size one or two. This should have been $C_n = C_{n-1} + (n-1)C_{n-2}$ for $n \geq 2$, and $C_0 = 1 = C_1$.

(a) Use the recurrence relation for C_n and induction on n to prove that:

(i) C_n is even for all $n \geq 2$;

(ii) $C_n \geq \sqrt{n!}$ for all $n \geq 0$.

(b) Show that the exponential generating function $\phi(t) = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!}$ satisfies the functional equation

$$\frac{d\phi}{dt} = (1+t)\phi(t).$$