MTH6109 Combinatorics

Assignment 3 — For handing in on 20 October 2011

Your solution to a selection of questions of your own choice should be handed in to the RED box on the GROUND floor by 3.30pm on Thursday.

Include with your solution a brief *self-assessment*, explaining how well you have understood the week's work, any areas of particular difficulty, why you have chosen to hand in these particular question(s), how well you think you have answered it, and (important:) what help you had with it.

I will mark a reasonable amount of your solutions, and hand back marked work (and discuss it with you, where necessary) at the tutorials the following week.

The learning objectives for this week are: being able to derive recurrence relations in certain cases; solving linear recurrence relations with constant coefficients.

If these questions are too easy, then there are some more challenging questions on last year's exercises which you can find on the module web-site.

1 Let C_n be the number of partitions of a set of size n into subsets of size one or two.

(a) Calculate C_1, C_2 and C_3.

(b) Derive a recurrence relation for C_n.

2 Solve the following recurrence relations.

(a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, and $a_0 = 1$, $a_1 = 3$.

(b) $b_n = 6b_{n-1} - 9b_{n-2}$ for $n \geq 2$, and $b_0 = 1$, $b_1 = 6$.

(c) $c_n = \frac{n}{n-1}c_{n-1}$ for $n \geq 2$, and $c_1 = 2$.

3 Solve the recurrence relation and intial conditions

$$a_0 = 2, a_1 = 4, a_2 = 7, \quad a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3} \text{ for } n \geq 3.$$

4 I purchase an item costing n pence. I have a large number of 1 and 2 pence coins at my disposal. In how many ways can I pay for the item

(a) if I an buying it from a machine and have to insert the coins one at a time;

(b) if I am buying it in a shop and can hand the money over all at once?

5 Solve the recurrence relation and initial conditions

$$a_0 = 1, \qquad a_1 = 1, \qquad a_n = 3a_{n-1} - 2a_{n-2} \text{ for } n \geq 2.$$

6 Solve the recurrence relation and initial conditions

$$a_0 = 2, \qquad a_n = (a_{n-1})^2 \text{ for } n \geq 1.$$

7 Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Prove by induction that

$$A^{n+1} = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

for $n \geq 1$, where F_n is the nth Fibonacci number.

8 Let c_n denote the number of ways of bracketing an expression $x_1.x_2.\cdots.x_n$, so that each pair of brackets contains just two expression to multiply together. (Thus $c_2 = 1$ and $c_3 = 2$, because there are two possible bracketings $(x_1x_2)x_3$ and $x_1(x_2x_3)$.)

Show that

$$c_n = \sum_{i=1}^{n-1} c_i c_{n-i},$$

and hence calculate c_4, c_5, $(\ldots)$.