Queen Mary
University of London

MTH6109 Combinatorics

Assignment 2 — For handing in on 13 October 2011

Your solution to a selection of questions of your own choice should be handed in to the RED box on the GROUND floor by 3.30pm on Thursday.

Include with your solution a brief *self-assessment*, explaining how well you have understood the week's work, any areas of particular difficulty, why you have chosen to hand in these particular question(s), how well you think you have answered it, and (important:) what help you had with it.

I will mark a reasonable amount of your solutions, and hand back marked work (and discuss it with you, where necessary) at the tutorials the following week.

The learning objectives for this week are: use counting methods to prove binomial identities; solve equations of the form $\sum_{i=1}^{k} x_i = n$; counting permutations of various types.

If these questions are too easy, then there are some more challenging questions on last year's exercises which you can find on the module web-site.

1 Write 1001 as a binomial coefficient $\binom{n}{k}$ with $n \leq 20$.

2 If X is a set of 8 elements, then the number of 3-element subsets of X is twice the number of 2-element subsets. Is there any other number for which this holds?

3 Use the binomial theorem to prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

Deduce that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}.$$

4 (a) How many permutations of $\{1,\dots,9\}$ are there?

(b) How many of them consist of a single cycle?

(c) How many of them have exactly three cycles, none of which is of length 1?

5 (a) In how many ways can 25 sweets be distributed to a class of 12 children?

(b) How many ways are there if each child is to have at least one sweet?

(c) How many ways are there if each child is to have at least two sweets?

6 (a) Calculate $\sum_{k=0}^{n} k\binom{n}{k}$.

(b) Calculate $\sum_{k=0}^{n} k^2\binom{n}{k}$.

7 (a) Prove that

$$\binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}.$$

(b) Prove that, if $n > 2k+1$, then $\binom{n}{k+1} > \binom{n}{k}$; if $n = 2k+1$, then $\binom{n}{k+1} = \binom{n}{k}$; and if $n < 2k+1$, then $\binom{n}{k+1} < \binom{n}{k}$.

(c) Hence show that, for fixed n and $k = 0,1,\dots,n$, the binomial coefficients increase, then remain constant for one step (if k is odd), then decrease. (Such a sequence is said to be *unimodal*.)

(d) Show further that the largest binomial coefficient is $\binom{2m}{m}$ if $n = 2m$ is even, while if $n = 2m+1$ is odd, then $\binom{2m+1}{m}$ and $\binom{2m+1}{m+1}$ are equal largest.

(e) Deduce that, if $n = 2m$, then

$$\frac{2^{2m}}{2m+1} \leq \binom{2m}{m} \leq 2^{2m}.$$