Queen Mary University of London

MTH6109 Combinatorics

Assignment 10 — For handing in on 15 December 2011

As usual, hand in a selection of questions of your choice to the red box on the ground floor by 3.30pm on Thursday 15th December.

The topics of this sheet are (a) intersecting families and (b) Sperner families.

1 Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. Construct three different families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of distinct subsets of X such that $|\mathcal{F}_i| = 7$ and, for all distinct $A_j, A_h \in \mathcal{F}_i$, we have $|A_j \cap A_h| = 1$, for all $i \in \{1, 2, 3\}$, and such that:

(a) $\{2\} \in \mathcal{F}_1$;

(b) $\{1, 2, 3, 5, 6, 7\} \in \mathcal{F}_2$;

(c) $\{1, 3, 5\} \in \mathcal{F}_3$.

2 Let $\mathcal{F}$ be an intersecting family of 2-element subsets of $\{1, \ldots, n\}$. Show that *either*

(a) there is an element $x \in \{1, \ldots, n\}$ contained in every set in $\mathcal{F}$, *or*

(b) $\mathcal{F} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$ for some $x, y, z \in \{1, \ldots, n\}$.

3 Let $X = \{1, \ldots, n\}$. Show that, for every non-empty subset A of X, there is an intersecting family $\mathcal{F}$ of subsets of X of size 2^{n-1} with $A \in \mathcal{F}$. Show further that any two subsets A, B with $A \cap B \neq \emptyset$ are contained in a family with these properties. What about three pairwise intersecting sets?

4 Show that the largest Sperner family of subsets of $\{1, 2, 3, 4, 5\}$ containing the sets $\{1, 2\}$ and $\{3, 4, 5\}$ contains eight sets. How does this compare with Sperner's Theorem?

5 Let X be a set with n elements and $\mathcal{F}$ be a Sperner family of distinct subsets of X. We showed in lectures that $|\mathcal{F}| \leq \binom{n}{m}$ where $m = \lfloor n/2 \rfloor$. Adapt the proof to show that if n is *even* and $|\mathcal{F}| = \binom{n}{m}$ then we must have

$$\mathcal{F} = \{A \subseteq X \ : \ |A| = m\}.$$

6 Let $\mathcal{S}$ be the family

$$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$$

of subsets of $\{1,2,3,4,5,6,7\}$.

Let $\mathcal{F}$ be the family of all subsets of $\{1,\ldots,7\}$ which contain a member of $\mathcal{S}$. Show that

(a) $\mathcal{F}$ is an intersecting family;

(b) $\mathcal{F}$ contains 7 sets of size 3, 28 of size 4, 21 of size 5, 7 of size 6, and 1 of size 7; in all, 64 sets.

7 Let $X = \{1,2,3,...,2k\}$.

(a) Construct an intersecting family $\mathcal{F}$ of distinct subsets of X such that $|A| = k$ for all $A \in \mathcal{F}$ and $|\mathcal{F}| = \binom{2k-1}{k-1}$.

(b) Prove that if $\mathcal{F}$ is an intersecting family of distinct subsets of X such that $|A| = k$ for all $A \in \mathcal{F}$ then $|\mathcal{F}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$.