Combinatorics Assignment 6: Solutions

1. (a) We are given the set of k positions which are coloured red, so we have to choose whether to colour each of the additional $n-k$ elements blue or green. There are two choices for each, and the choices are independent, so the number is 2^{n-k}.

(b) Now we are only given that there are k red elements but not told their positions. There are $\binom{n}{k}$ ways to choose the positions of the red elements, and (by part (a)) 2^{n-k} ways of colouring the remaining elements. So

$$A_k = \binom{n}{k} 2^{n-k}.$$

(c) The generating function is

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} x^k = (x+2)^n,$$

where we have used the Binomial Theorem in the last step.

The total number of colourings is $\sum_{k=0}^{n} A_k$, which is what we get when we substitute $x = 1$ into the generating function. So the total number is $(1+2)^n = 3^n$.

(d) This is the total number of ways of colouring the n elements red, green or blue. But we can simply observe that there are 3 choices for the colour of each element, and the choices are independent.

2. (a) You cannot partition an odd number of objects into sets of size 2; so $f_n = 0$ if n is odd.

For n even, we proceed by induction. Clearly $f_2 = 1$, so the induction starts. Now suppose that the result is true for $n-2$. (We go in steps of 2 in the induction.) Now given the numbers $1, \ldots, n$, consider the number n; there are $n-1$ choices for the number which is paired with n, and once we have chosen this number, we have to partition the remaining $n-2$ objects into sets of size 2. So

$$f_n = (n-1) f_{n-2},$$

from which the required formula follows by induction.

(b) For $n = 2m$, $n!$ is the product of all the integers from 1 to n. Now

$$2^m\, m! = (2 \cdot 1)(2 \cdot 2) \cdots (2 \cdot m)$$

is the product of the even numbers; so the quotient is the product of the odd numbers.

(If you prefer, you can write the right-hand side out as a product and cancel factors to get the result.)

Note: You can also prove this directly, and then "reverse engineer" (a) from (b). How do we partition the set into parts of size 2? We take m boxes, each with room for two numbers, and we put the numbers $1, \ldots, 2m$ into the boxes (which we can do in $(2m)!$ ways). But we get the same partition if (i) we put the boxes in a different order (there are $m!$ orders for the boxes), or if (ii) we put the two elements in each box in a different order (there are 2 orders for each box, and so 2^m altogether).

(c) Now

$$\begin{aligned}
\sum_{n\geq 0} \frac{f_n x^n}{n!} &= \sum_{m\geq 0} \frac{(2m)! x^{2m}}{(2m)! 2^m m!} \\
&= \sum_{m\geq 0} \frac{x^{2m}}{2^m m!} \\
&= \sum_{m\geq 0} \frac{(x^2/2)^m}{m!} \\
&= \mathrm{e}^{x^2/2}.
\end{aligned}$$

3 (a) Given a permutation on $\{1, \ldots, n\}$, its cycle decomposition gives us a partition of $\{1, \ldots, n\}$. Hence there is a map from permutations with k cycles to partitions with k parts. This map is onto – for every partition, we can construct a permutation which has the parts of the partition as its cycles – though not in general one-to-one. So there are at least as many permutations as partitions.

(b) $\displaystyle\sum_{k=0}^{n} |s(n,k)|$ and $\displaystyle\sum_{k=0}^{n} S(n,k)$ simply count the total number of permutations of $\{1, \ldots, n\}$, or partitions of $\{1, \ldots, n\}$, respectively; these numbers are respectively $n!$ and the Bell number $B(n)$.

Remark: From parts (a) and (b) you deduce that $B(n) \leq n!$, which was the content of an exercise on Assignment 4.

(c) Use the recurrence relations:

$$\begin{aligned} S(n,1) &= 1, & S(n,n) &= 1, \\ S(n,k) &= S(n-1,k-1) + kS(n-1,k) \quad \text{for} & 1 &< k < n \end{aligned}$$

and

$$\begin{aligned} s(n,1) &= (-1)^{n-1}(n-1)!, & s(n,n) &= 1, \\ s(n,k) &= s(n-1,k-1) - (n-1)s(n-1,k) \quad \text{for} & 1 &< k < n. \end{aligned}$$

The required tables are, for $S(n,k)$,

$$\begin{array}{ccccc} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 7 & 6 & 1 \\ 1 & 15 & 25 & 10 & 1 \end{array}$$

and for $s(n,k)$,

$$\begin{array}{ccccc} 1 \\ -1 & 1 \\ 2 & -3 & 1 \\ -6 & 11 & -6 & 1 \\ 24 & -50 & 35 & -10 & 1 \end{array}$$

Now multiply the matrices and check that the identity is obtained.

4. Here is one way to solve this problem. It depends on the following fact (check that you can see why this should be true):

> Let $(x_1, x_2, \ldots, x_n)$ be a list of real numbers and y a real number. Suppose that (at least) q elements of the list are greater than y. Then, if we arrange the list in decreasing order, the largest q elements of the list are all greater than y.

Let a_{ij} be the height of the soldier in row i and column j *after the first rearrangement*. The rows are in decreasing order, so if $j < k$, then $a_{ij} > a_{ik}$ for $i = 1, \ldots, m$.

We are going to consider just columns j and k, so to simplify the notation we will let $b_i = a_{ij}$ (the height of the ith soldier in column j) and $c_i = a_{ik}$

(the height of the ith soldier in column k). Now we are given that $b_i > c_i$ for $i = 1, \ldots, m$.

We look at the first row before going on to the general case. After the second rearrangement, the soldiers in the first row in columns j and k have heights which are the greatest elements of the lists $(b_1, \ldots, b_m)$ and $(c_1, \ldots, c_m)$ respectively. Let the greatest element of the second list be c_{i_1}. Then at least one element of the b list, namely b_{i_1}, is greater than c_{i_1}; hence by our observation, the greatest element of the b list is greater than c_{i_1}.

Now we do the general case. Let c_{i_s} be the sth greatest element of the c list. (The soldier with this height will go into the sth row after the second rearrangement). Then there are at least s elements of the b list which are greater than c_{i_s}, namely $b_{i_1}, b_{i_2}, \ldots, b_{i_s}$: for $b_{i_s} > c_{i_s}$ by assumption, and if $t < s$ then $b_{i_t} > c_{i_t} > c_{i_s}$. So, by our observation, the s largest elements of the b list are all greater than c_s. These are the heights of the soldiers in rows $1, \ldots, s$ of column j after the second rearrangement. In particular, the soldier in row s of column j has height greater than c_{i_s}, the soldier in row s of column k, after the second rearrangement.

So the theorem is proved.