Queen Mary
University of London

MTH6109 Combinatorics

Assignment 6 For handing in on 17 November 2011

Your solution to a selection of questions of your own choice should be handed in to the RED box on the GROUND floor by 3.30pm on Thursday.

Include with your solution a brief *self-assessment*, explaining how well you have understood the week's work, any areas of particular difficulty, why you have chosen to hand in these particular question(s), how well you think you have answered it, and (important:) what help you had with it.

I will mark a reasonable amount of your solutions, and hand back marked work (and discuss it with you, where necessary) at the tutorials the following week.

The learning objectives for this week are: further understanding of generating functions (being able to swith between a generating function and a formula for its coefficients), and understanding of Stirling numbers and their relationship.

If these questions are too easy, then there are some more challenging questions on last year's exercises which you can find on the module website.

1 We want to count the number of ways in which the numbers $1,2,\ldots,n$ can be coloured red, green and blue; we are particularly interested in the number of red elements.

(a) Suppose that we specify a set of k red numbers, the rest being green and blue. Show that there are 2^{n-k} such colourings.

(b) Let A_k be the total number of colourings with k red elements. Show that

$$A_k = \binom{n}{k} 2^{n-k},$$

and dedce that the generating function for the numbers A_k is

$$\sum_{k=0}^{n} A_k x^k = (x+2)^n.$$

(c) Deduce that there are 3^n colourings altogether.

(d) Prove the last assertion directly.

2 Let f_n be the number of ways of partitioning the set $\{1,\ldots,n\}$ into parts of size 2.

(a) Prove that

$$f_n = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ 1\cdot 3\cdot 5\cdots(n-1) & \text{if } n \text{ is even.} \end{cases}$$

(b) Show that, if $n = 2m$, then $f_n = (2m)!/2^m m!$.

(c) Hence show that

$$\sum_{n\geq 0} \frac{f_n x^n}{n!} = \mathrm{e}^{x^2/2}.$$

3 Recall that the Stirling number of the second kind, $S(n,k)$, counts the partitions of $\{1,\ldots,n\}$ into k parts, while the Stirling number of the first kind, $s(n,k)$, is $(-1)^{n-k}$ times the number of permutations of $\{1,\ldots,n\}$ with k cycles.

(a) Prove that $S(n,k) \leq |s(n,k)|$. [Hint: How does a permutation of $\{1,\ldots,n\}$ give rise to a partition of this set?]

(b) Prove that $\sum_{k=1}^{n} |s(n,k)| = n!$ and $\sum_{k=1}^{n} S(n,k) = B(n)$, where $B(n)$ is the h-th Bell number.

(c) Write down tables of the values of $S(n,k)$ and $s(n,k)$ for $1 \leq n,k \leq 5$, and verify that, regarded as matrices, they are inverses of each other.

4 This is a "bonus question", and is not related to any particular piece of course material. It is not straightforward, but you may enjoy the challenge of solving it.

A platoon of soldiers is arranged in a rectangular array on the parade ground. The platoon sergeant arranges the soldiers in each row in decreasing order of height. Then he arranges the soldiers in each column in decreasing order of height. Show that he does not need to arrange the soldiers in each row again (that is, after the second rearrangement, the rows are still in decreasing order of height.)