MTH6109 Combinatorics

Solutions 3 November 2011

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1 (a) $C_1 = 1$, $C_2 = 2$, $C_3 = 4$;

(b) Let the set be $\{1, 2, \ldots, n\}$, and look at the part of the partition containing n. Either this has size 1, and then there are C_{n-1} ways of partitioning the remaining points; or it has size 2, so is $\{x, n\}$ for some x (so $n-1$ choices), and there are C_{n-2} ways of partitioning the remaining points. Therefore

$$C_n = C_{n-1} + (n-1)C_{n-2}.$$

2 (a) The characteristic equation is $x^2 - x - 6 = 0$, with solutions $x = 3$, $x = -2$, so the general solution is $a_n = A.3^n + B.(-2)^n$. Substituting $n = 0$ and $n = 1$ gives $A + B = 1$ and $3A - 2B = 3$, which have solution $A = 1, B = 0$. Hence $a_n = 3^n$.

(b) The characteristic equation is $x^2 - 6x + 9 = 0$, i.e. $(x-3)^2 = 0$. Hence the genral solution is $b_n = (A + Bn).3^n$. Substituting $n = 0, 1$ we get $A = 1$, $3(A + B) = 6$, so $B = 1$ and the solution is $b_n = (1 + n)3^n$.

(c) This recurrence relation does not have constant coefficients, so the above method does not work. But if you calculate a few values you will soon realise that $c_n = 2n$. Now you can prove it by induction. It is true for $n = 1$, since $c_1 = 2$, and if $c_{n-1} = 2(n-1)$, then

$$c_n = \frac{n-1}{n}.2(n-1) = 2n.$$

3 The characteristic equation is $x^3 - 4x^2 + 5x - 2 = 0$ and it is easy to see that $x = 1$ is a root, so we have

$$x^3 - 4x^2 + 5x - 2 = (x-1)(x^2 - 3x + 2) = (x-1)^2(x-2).$$

Thus the general solution is $a_n = A + Bn + C.2^n$. Substituting in $n = 0, 1, 2$ gives the equations $A + C = 2$, $A + B + 2C = 4$ and $A + 2B + 4C = 7$, and solving these in the usual way gives $A = B = C = 1$, so $a_n = 1 + n + 2^n$.

4 (a) This was done in the notes at the beginning of Chapter 2. Suppose the number of ways is W_n. The first coin is either 1p or 2p. If it is 1p, there are W_{n-1} ways of paying the remaining $n-1$ pence. If it is 2p, there are W_{n-2} ways of paying the remaining $n-2$ pence. Hence $W_n = W_{n-1} + W_{n-2}$. We also have the initial conditions $W_1 = 1$ and $W_2 = 2$, so these are the Fibonacci numbers.

(b) The number of 2p coins used is either 0, 1, 2,..., or $\lfloor n/2 \rfloor$, so the total number of possibilities is $\lfloor n/2 \rfloor + 1$.

5 Easy.

6 Claim $a_n = 2^{2^n}$. Proof by induction. For $n = 0$ we have $2^{2^0} = 2^1 = 2 = a_0$. If $a_{n-1} = 2^{2^{n-1}}$ then $a_n = (a_{n-1})^2 = (2^{2^{n-1}})^2 = 2^{2^{n-1}.2} = 2^{2^n}$.

7 Routine.

8 The outermost bracketing must be as

$$(x_1.\cdots.x_i).(x_{i+1}.\cdots.x_n)$$

for some $i = 1, 2, \ldots, n-1$. Now there are c_i ways to bracket the expression $x_1.\cdots.x_i$, and c_{n-i} ways to bracket the expression $x_{i+1}.\cdots.x_n$, so by the multiplication principle, there are $c_i c_{n-i}$ ways to complete the bracketing. By the addition principle, we need to sum this over all values of i, giving

$$c_n = \sum_{i=1}^{n-1} c_i c_{n-i}.$$

Hence $c_4 = c_1c_3 + c_2c_2 + c_3c_1 = 2 + 1 + 2 = 5$ and $c_5 = c_1c_4 + c_2c_3 + c_3c_2 + c_4c_1 = 5 + 2 + 2 + 5 = 14$.