MTH5112 Linear Algebra I 2012–2013

Coursework 9

Please hand in your solutions of the **starred** feedback exercises by **noon on Friday 7 December** using the red Linear Algebra I Collection Box in the Basement. Don't forget to put your **name** (with your surname underlined) and **student number** on your solutions, and to **staple** them.

Exercise 1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be the basis of $\mathbb{R}^3$ given in Exercise 2 on Coursework 7, that is,

$$\mathbf{b}_1 = (1, -2, 0)^T, \quad \mathbf{b}_2 = (0, 1, 1)^T, \quad \mathbf{b}_3 = (-3, 6, 1)^T,$$

and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$L(\mathbf{x}) = (3x_1 + 3x_2 + 3x_3, -4x_1 - 5x_2 + 6x_3, 6x_1 + 3x_2 - 2x_3)^T.$$

(a) Find the matrix representation of L with respect to the standard basis of $\mathbb{R}^3$.

(b) Using (a) and the results from Exercise 4 (b) on Coursework 7, determine the matrix representation of L with respect to $\mathcal{B}$.

Exercise* 2. Let $L : P_2 \to P_2$ be the linear transformation given by

$$(L(\mathbf{p}))(t) = 2t\mathbf{p}(0) + \mathbf{p}(1 + 2t).$$

Let $\mathcal{P} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ and $\mathcal{Q} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ be the bases of P_2 given by

$$\mathbf{p}_1(t) = 1, \quad \mathbf{p}_2(t) = t, \quad \mathbf{p}_3(t) = t^2, \quad \text{and} \quad \mathbf{q}_1(t) = 1-t, \quad \mathbf{q}_2(t) = 1+2t, \quad \mathbf{q}_3(t) = 3+7t+2t^2.$$

(a) Find the matrix representation of L with respect to $\mathcal{P}$.

(b) Find the transition matrix S from $\mathcal{Q}$ to $\mathcal{P}$ and the transition matrix from $\mathcal{P}$ to $\mathcal{Q}$.

(c) Using (a) and (b), determine the matrix representation of L with respect to $\mathcal{Q}$.

(d) If $\mathbf{p} \in P_2$ is given by $\mathbf{p}(t) = c_1(1 - t) + c_2(1 + 2t) + c_3(3 + 7t + 2t^2)$ for some $c_1, c_2, c_3 \in \mathbb{R}$ and n is a positive integer, find $L^n(\mathbf{p})$.

Exercise 3. Prove the Pythagorean Theorem in $\mathbb{R}^n$, that is, show that two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are orthogonal if and only if

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

Exercise 4. Let H be a subspace of $\mathbb{R}^n$. Show that $H^\perp$ is a subspace of $\mathbb{R}^n$.

Exercise* 5. Let H be the subspace of $\mathbb{R}^3$ spanned by the two vectors $\mathbf{y} = (1, -1, 1)^T$ and $\mathbf{z} = (0, 1, -3)^T$.

(a) Find a basis of $H^\perp$. [Hint: $H^\perp$ is the nullspace of a 2×3 matrix.]

(b) Give a geometric description of H and $H^\perp$.

Exercise 6. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in $\mathbb{R}^n$ and let $H = \mathrm{Span}\,(\mathbf{v}_1, \ldots, \mathbf{v}_r)$. Show that $\mathbf{x} \in H^\perp$ if and only if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, r$.

Exercise 7. Let $A \in \mathbb{R}^{m \times n}$ and let $\mathbf{x}$ be in the column space of A. If $A^T\mathbf{x} = \mathbf{0}$ what is $\mathbf{x}$? [Hint: Use the Fundamental Subspace Theorem.]