MTH5112 Linear Algebra I 2012–2013

Coursework 8

Please hand in your solutions of the **starred** feedback exercises by **noon on Friday 30 November** using the red Linear Algebra I Collection Box in the Basement. Don't forget to put your **name** (with your surname underlined) and **student number** on your solutions, and to **staple** them.

Exercise 1. Determine which of the following are linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$:

(a) $L(\mathbf{x}) = (x_2 + x_3, x_3 + x_1)^T$, (b) $L(\mathbf{x}) = (x_1 + x_2, 1)^T$,
(c) $L(\mathbf{x}) = (x_2 - 2x_1, x_3)^T$, (d) $L(\mathbf{x}) = (x_1^2, x_2)^T$.

Exercise 2. Let C be a fixed $n \times n$ matrix. Determine which of the following are linear transformations from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^{n\times n}$:

(a) $L(A) = 5A$, (b) $L(A) = AC + CA$,
(c) $L(A) = ACA$, (d) $L(A) = CAC$.

Exercise 3. Determine which of the following are linear transformations from P_2 to P_2:

(a) $(L(\mathbf{p}))(t) = t^2\mathbf{p}(0)$, (b) $(L(\mathbf{p}))(t) = \mathbf{p}'(t) - t$.

Exercise* 4. Let M be a fixed $n \times n$ matrix with $M \neq O$. Determine which of the following are linear transformations and justify your answers.

(a) $L : \mathbb{R}^2 \to \mathbb{R}^3$, $L(\mathbf{x}) = (1 + x_1, 2 + x_2, 3)^T$; (b) $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L(\mathbf{x}) = (x_3 - x_2, x_2 - x_1)^T$;
(c) $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $L(A) = A^T$; (d) $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $L(A) = AM^2$;
(e) $L : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $L(A) = A^2M$; (f) $L : C[0,1] \to C[0,1]$, $(L(\mathbf{f}))(t) = 2t$.

Exercise 5. A linear transformation $L : V \to W$ is said to be **one-to-one** if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$ (that is, no distinct vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ get mapped to the same vector $\mathbf{w} \in W$). Show that L is one-to-one if and only if $\ker(L) = \{\mathbf{0}\}$.

Exercise 6. For each of the following linear transformations find its matrix representation with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$:

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1 - 2x_3, x_1 + 7x_2)^T$;

(b) $L : \mathbb{R}^3 \to \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2 + x_3, 5x_1 - x_3, x_2 - 4x_1)^T$;

(c) $L : \mathbb{R}^2 \to \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2, x_1, x_1 - x_2)^T$.

Exercise* 7. Let $L : P_2 \to P_2$ be the mapping given by

$$(L(\mathbf{p}))(t) = \mathbf{p}(t-1),$$

and let $\mathcal{P} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be the standard basis of P_2 given by

$$\mathbf{p}_1(t) = 1, \quad \mathbf{p}_2(t) = t, \quad \mathbf{p}_3(t) = t^2.$$

(a) Show that L is a linear transformation.

(b) Find the matrix representation of L with respect to $\mathcal{P}$.