MTH5112 Linear Algebra I 2012–2013

Coursework 7

Please hand in your solutions of the **starred** feedback exercises by **noon on Friday 23 November** using the red Linear Algebra I Collection Box in the Basement. Don't forget to put your **name** (with your surname underlined) and **student number** on your solutions, and to **staple** them.

Exercise* 1. Let H be the subspace of $\mathbb{R}^4$ given by

$$H = \left\{ (r,s,t,u)^T \mid r,s,t,u \in \mathbb{R},\ r - 2s + t + 3u = 0 \text{ and } s + t - 4u = 0 \right\} .$$

Find a basis for H and determine $\dim H$.

Exercise 2. Consider the following vectors in $\mathbb{R}^3$:

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} -3 \\ 6 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{d}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \mathbf{d}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} .$$

Show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ are bases for $\mathbb{R}^3$.

Exercise* 3. Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in P_2$ be given by

$$\mathbf{p}_1(t) = 2 - 4t + t^2 , \quad \mathbf{p}_2(t) = 3 + t , \quad \mathbf{p}_3(t) = 1 .$$

(a) Show that $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for P_2.

(b) Suppose that $\mathbf{p} \in P_2$ has $\mathcal{B}$-coordinate vector $[\mathbf{p}]_{\mathcal{B}} = (-1, 3, 2)^T$. Find $\mathbf{p}$.

(c) Let $\mathbf{q} \in P_2$ be given by $\mathbf{q}(t) = 6 - t^2$. Find the coordinates of $\mathbf{q}$ relative to $\mathcal{B}$.

Exercise 4. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ be the bases given in Exercise 2.

(a) What is the transition matrix S_1 from $\mathcal{D}$ to the standard basis?

(b) Determine the transition matrix S_2 from the standard basis to $\mathcal{B}$, and, hence, determine the transition matrix from $\mathcal{D}$ to $\mathcal{B}$.

(c) If the $\mathcal{D}$-coordinate vector of $\mathbf{x} \in \mathbb{R}^3$ is $(1, -3, 2)^T$, what is the $\mathcal{B}$-coordinate vector of $\mathbf{x}$?

Exercise 5. Let $A \in \mathbb{R}^{m \times n}$. Show the following: the system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if the column vectors of A are linearly independent.

Exercise 6. Let

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 2 \\ 2 & -2 & 6 & 3 & 0 \\ 3 & -3 & 9 & 4 & 2 \end{pmatrix} .$$

Find bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$ and $N(A)$. Determine the rank and nullity of A, and verify that the Rank-Nullity Theorem holds for the matrix A.

Exercise 7. Let $A \in \mathbb{R}^{n \times n}$. Use the Rank-Nullity Theorem to show the following: A is invertible if and only if $\operatorname{rank} A = n$.

Exercise 8. Is it possible to construct matrices with the following properties?

(a) A 4×3 matrix A with $\operatorname{rank} A = 1$ and $\operatorname{nul} A = 2$.

(b) A 3×4 matrix B with $\operatorname{rank} B = 2$ and $\operatorname{nul} B = 1$.

If yes, give an example; if not, explain why such a matrix cannot exist.