MTH5112 Linear Algebra I 20012–2013

Coursework 11

Please do not submit solutions to this final coursework, as it is not assessed and will not be marked. Solutions will be available from the course web site.

Exercise 1. Let $A \in \mathbb{R}^{m\times n}$.

(a) Show that $N(A^T A) = N(A)$.

(b) Use (a) to show that the columns of A are linearly independent if and only if $A^T A$ is invertible.

Exercise 2. Let A be a square matrix. Show the following:

(a) 0 is an eigenvalue of A if and only if A is not invertible;

(b) If A is invertible, then λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1};

(c) If λ is an eigenvalue of A and n a positive integer, then λ^n is an eigenvalue of A^n.

Exercise 3. Let

$$A = \begin{pmatrix} 4 & 3 & 3 \\ -4 & -3 & -4 \\ 2 & 2 & 3 \end{pmatrix} .$$

(a) Show that $(3, -4, 2)^T$ is an eigenvector of A and find the corresponding eigenvalue.

(b) Determine all eigenvalues of A and find bases for the corresponding eigenspaces.

(c) Using (b), explain why A is diagonisable and find a matrix P that diagonalises A.

(d) Using (c), find A^5.

Exercise 4. For each of the following matrices A, find the eigenvalues and find a basis for the corresponding eigenspaces. Decide whether the matrix is diagonalisable, and, if it is, find an invertible matrix P such that $P^{-1}AP$ is diagonal:

$$\text{(a)} \quad \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}, \qquad \text{(b)} \quad \begin{pmatrix} 6 & 4 & 2 \\ -7 & -6 & -5 \\ 4 & 4 & 4 \end{pmatrix}, \qquad \text{(c)} \quad \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \qquad \text{(d)} \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Exercise 5. Let A be a symmetric matrix.

(a) Show that $(A\mathbf{x})\boldsymbol{\cdot}\mathbf{y} = \mathbf{x}\boldsymbol{\cdot}(A\mathbf{y})$ for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(b) Use (a) to show that eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Exercise 6. Show the following converse of the Spectral Theorem: if a square matrix A is diagonalised by an orthogonal matrix, then A is symmetric. [Hint: use the fact that a diagonal matrix is symmetric.]

Exercise 7. For each of the following symmetric matrices A find an orthogonal matrix that diagonalises it, that is, find an orthogonal matrix Q such that $Q^T AQ = D$, where D is diagonal:

$$\text{(a)} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \text{(b)} \quad \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}, \qquad \text{(c)} \quad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

[Hint for (c): one of the eigenvalues is -2.]

Have a nice break!