MTH5112 Linear Algebra I 20012–2013

Coursework 10

Please hand in your solutions of the **starred** feedback exercises by **12 noon on Friday 14 December** using the red Linear Algebra I Collection Box in the Basement. Don't forget to put your **name** (with your surname underlined) and **student number** on your solutions, and to **staple** them.

Exercise 1. Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Show the following:

(a) $(U\mathbf{x})\cdot(U\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$;

(b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$;

(c) $U\mathbf{x}$ and $U\mathbf{y}$ are orthogonal if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Exercise* 2. Let $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. Show the following:

(a) Q is invertible and $Q^{-1} = Q^T$.

(b) If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then $\{Q\mathbf{v}_1, \dots, Q\mathbf{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. [Hint: use Exercise 1]

Exercise 3. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be the vectors in $\mathbb{R}^3$ given by

$$\mathbf{u}_1 = (1, 0, -3)^T, \quad \mathbf{u}_2 = (3, 1, 1)^T, \quad \mathbf{u}_3 = (-3, 10, -1)^T .$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Express $\mathbf{y} = (6, -8, 12)^T$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

Exercise 4. Let H be a subspace of $\mathbb{R}^n$. Show the following:

(a) $(H^\perp)^\perp = H$;

(b) $\dim H + \dim H^\perp = n$.

Exercise 5. Let $\mathbf{y} = (-1, 7)^T$ and $\mathbf{u} = (1, 3)^T$. Let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto $\mathrm{Span}\,(\mathbf{u})$. Find $\hat{\mathbf{y}}$. Calculate $\mathbf{y} - \hat{\mathbf{y}}$ and verify that $\mathbf{u}$ and $\mathbf{y} - \hat{\mathbf{y}}$ are orthogonal. Give a geometric description of the quantity $\|\mathbf{y} - \hat{\mathbf{y}}\|$.

Exercise 6. Let $\mathbf{y} = (6, -1, 8)^T \in \mathbb{R}^3$ and let H be the subspace of $\mathbb{R}^3$ spanned by the two vectors $\mathbf{u}_1 = (1, 2, 1)^T$ and $\mathbf{u}_2 = (-3, 1, 1)^T$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set and write $\mathbf{y}$ as a sum of a vector in H and a vector orthogonal to H.

Exercise* 7. Let H be the span of the three linearly independent vectors

$$\mathbf{x}_1 = (1, 0, 1, 0)^T, \quad \mathbf{x}_2 = (3, 0, 1, 1)^T, \quad \mathbf{x}_3 = (-2, 1, 4, -3)^T .$$

(a) Use the Gram Schmidt process to determine an orthogonal basis for H.

(b) Use your results from (a) to write $\mathbf{y} = (2, 1, 4, -4)^T$ as a sum of a vector in H and a vector orthogonal to H and determine the closest point to $\mathbf{y}$ in H.

Exercise 8. Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix} .$$

Write down the corresponding normal equations and determine the set of least squares solutions.