Chapter 2: Matrix Algebra

The main point of this chapter, apart from laying the foundations for most of the rest of the course, is to study <u>inverse matrices</u> and their ~~uses~~, uses, for example in <u>solving systems of linear equations.</u>

2.1 Revision (Chapter 7 of Geometry I)

<u>Terminology</u> An $m \times n$ <u>matrix</u> is a rectangular array of <u>scalars</u> (real numbers)

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

written $(a_{ij})_{m \times n}$ or $A = (a_{ij})$

The <u>size</u> of A is <u>$m \times n$</u>.

If $m = n$ the matrix is <u>square</u>.

Two matrices $A = (a_{ij})$ & $B = (b_{ij})$ are <u>equal</u> if they have the same size <u>and</u> every ~~$a_{ij} = b_{ij}$~~

$$a_{ij} = b_{ij}$$

<u>Scalar multiplication</u> If α is a real number and $A = (a_{ij})$ is a matrix then $\alpha A = (\alpha a_{ij})$

<u>Addition</u> If $A = (a_{ij})$, $B = (b_{ij})$ are two matrices of the same size, we <u>define</u> $A + B = (a_{ij} + b_{ij})$

<u>Zero matrix</u> $0_{m \times n} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}$

With these definitions we have the following <u>facts</u>:

- $A + B = B + A$ (because $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ for all i, j)

- $(A+B)+C = A+(B+C)$
- $A+0 = A$
- $A+(-A) = 0$
- $\alpha(A+B) = \alpha A + \alpha B$
  (since $\alpha(a_{ij}+b_{ij}) = \alpha a_{ij} + \alpha b_{ij}$)
- $(\alpha+\beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = (\alpha\beta)A$
- $1 . A = A$.

Lecture 5

Just because I don't take an attendance register in tutorials, it does not mean attendance is optional!

Attendance	Wed:	$\sim 30/120$
	Thus. 11	$\sim 10/60$
	Thus 2	$\sim 30/60$

<u>Matrix multiplication</u> is defined as it is in order to <u>encode</u> row operations. (see later)

If $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{n \times p}$

then $C = AB$ is an $m \times p$ matrix

$$= (c_{ij})_{m \times p}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

<u>Example 2.10</u> Compute the $(1,3)$ entry of AB

where $A = \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array}\!\!\begin{array}{c} \begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \\ \underbrace{\;1 \quad 2 \quad 3\;}_{k} \end{array}$, $B = \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ \end{array}\!\!\begin{array}{c} \begin{pmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{pmatrix} \\ \underbrace{\;1 \quad 2 \quad 3 \quad 4\;}_{j} \end{array}$

i

Answer $= (3 \times 6) + (-1 \times 3) + (2 \times 5) = 18 - 3 + 10 = 25$. ~~whatever~~.

<u>Identity matrix</u> $I = I_n$ denotes $n \times n$ identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & & 0 & 1 \end{pmatrix}.$$

Properties of matrix multiplication:

- $IA = A$ (if A is $m \times n$, then I is $m \times m$)

- $BI = B$ (if B is $m \times n$ then I is $n \times n$).

- $(AB)C = A(BC)$ Not obvious:

Why? If A is $m \times n$
B is $n \times p$
C is $p \times q$

then let $D = AB = (d_{ij})_{m \times p}$

so that $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

& if $X = (AB)C = DC = (x_{ij})$

we have
$$x_{ij} = \sum_{l=1}^{p} d_{il} c_{lj}$$
$$= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj}$$
$$= \sum_{l=1}^{p} \left(\sum_{k=1}^{n} \left(\underline{(a_{ik} b_{kl}) c_{lj}} \right) \right)$$
$$= \sum_{l=1}^{p} \sum_{k=1}^{n} a_{ik} (b_{kl} c_{lj})$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{p} a_{ik}(b_{kl} c_{lj})$$

does not depend on l

$$= \sum_{k=1}^{n} a_{ik} \underbrace{\left(\sum_{l=1}^{p} b_{kl} c_{lj} \right)}_{\substack{(k,j) \text{ entry of } BC \\ = (e_{kj}) \text{ where } BC = E = (e_{ij}).}}$$

$$= \sum_{k=1}^{n} a_{ik} e_{kj}$$

$= (i,j)$ entry of $AE = A(BC)$.

- $A(B+C) = AB + AC$

$$\left(\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj} \right)$$

- $(B+C)A = BA + CA$.
- $(\alpha A)B = \alpha(AB) = A(\alpha B)$.

where α is any scalar

<u>Warning</u> In general $AB \neq BA$!

<u>Example</u> $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

exercise: show $AB \neq BA$.

In the unlikely situation that $AB = BA$ we say that A and B <u>commute</u>.

<u>Inverse matrices</u>

(Recall: if $x \in \mathbb{R}$ and $x \neq 0$, then x^{-1} is the real number such that $x . x^{-1} = 1$.) — & $x^{-1} . x = 1$

If A is an $n \times n$ matrix, and B is a $(n \times n)$ matrix such that

$$AB = I$$

and

$$BA = I$$

then B is called <u>the</u> <u>inverse</u> of A,

& we write $B = A^{-1}$.

In this case A is called <u>invertible</u>.

If there is no such matrix B, then A is <u>non-invertible</u>.

Important fact (2.17) If B and C are both inverses of A, then

$$B = C.$$

Reasons $AB = I$, $BA = I$,
$AC = I$, $CA = I$

so $B = BI = B(AC) = (BA)C =$

$= IC \qquad = C.$

Done

If A is invertible, with inverse A^{-1}, is A^{-1} invertible?

Well, $A \underline{\underline{A}}^{-1} = \underline{\underline{A}}^{-1} A = I$

so yes, A is the inverse of A^{-1}.

If A and B are invertible $n \times n$ matrices, is AB invertible?

Lecture no. 6

To show that $X = AB$ is invertible we need to find a matrix Z, say, Such that $XZ = I$ and $ZX = I$.

Try $Z = B^{-1}A^{-1}$.

Then
$$
\begin{aligned}
XZ &= (AB)(B^{-1}A^{-1}) \\
&= A(B(B^{-1}A^{-1})) \\
&= A((BB^{-1})A^{-1}) \\
&= A(IA^{-1}) \\
&= AA^{-1} \\
&= I
\end{aligned}
$$

and similarly $ZX = (B^{-1}A^{-1})(AB) = I$

So Z is the inverse of X.

Inverse matrices are "useful" for solving systems of linear equations.

2.2 Transpose of a matrix

Example $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

Definition If A is an $m \times n$ matrix,

$$A = (a_{ij})_{m \times n},$$

then the <u>transpose</u> of A, is

$$A^T = (b_{ij})_{n \times m}$$

where $b_{ij} = a_{ji}$.

Properties of transposed matrices

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(\alpha A)^T = \alpha A^T$
- $(AB)^T = B^T A^T$

<u>Why?</u> If $C = AB$

so that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

then $D = C^T$ has (i,j) entry

$$d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

Similarly ~~then~~ the (i,j) entry of $\underline{B}^T \underline{A}^T$ is

$$\sum_{k=1}^{n} f_{\underline{ik}} \, \underline{g}_{kj} \quad \text{where}$$

$B^T = (f_{ij})$ and $A^T = (g_{ij})$.

So $\underline{f_{ij} = b_{ji}}$ and $\underline{g_{ij} = a_{ji}}$

So (i,j) entry of $B^T A^T$ is

$$\sum_{k=1}^{n} b_{ki} a_{jk} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$\Rightarrow (AB)^T = B^T A^T.$$

If $A A^{-1} = I$ and $A^{-1} A = I$

then $(A A^{-1})^T = I^T = I$ & $(A^{-1}A)^T = I$

$\Rightarrow (A^{-1})^T A^T = I$ and $A^T (A^{-1})^T = I$.

So this is the inverse of A^T.

i.e. <u>Fact</u> A^T is invertible

and $(A^T)^{-1} = (A^{-1})^T$.

2.3 Types of square matrices

A (square) matrix A is called <u>symmetric</u>

if $A = A^T$.

A square matrix $A = (a_{ij})$ is called

- <u>upper triangular</u> if

$$a_{ij} = 0 \quad \text{whenever} \quad i > j$$

- <u>strictly upper triangular</u> if

$$a_{ij} = 0 \quad \text{whenever} \quad i \geq j$$

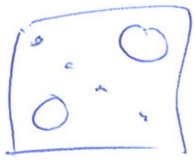
- <u>lower triangular</u> if

$$a_{ij} = 0 \quad \text{whenever} \quad i < j$$

- <u>strictly ~~triangular~~ lower triangular</u> if

$$a_{ij} = 0 \quad \text{whenever} \quad i \leq j$$

- <u>diagonal</u> if



$$a_{ij} = 0 \quad \text{whenever} \quad i \neq j.$$

Facts about triangular matrices

If A, B are upper triangular $n \times n$ matrices

what can we say about AB?

Special case of two diagonal matrices:

$$\begin{pmatrix} x_1 & 0 & 0 & \\ 0 & x_2 & & 0 \\ \vdots & 0 & \ddots & \\ & & & x_n \end{pmatrix} \begin{pmatrix} y_1 & 0 & & \\ & y_2 & & 0 \\ & 0 & \ddots & \\ & & & y_n \end{pmatrix} = \begin{pmatrix} x_1 y_1 & 0 & \cdots & 0 \\ & \ddots & & \\ & & & x_n y_n \end{pmatrix}$$

The product of two diagonal matrices is diagonal.

Exercise Prove this direct from the definition.

Fact If A & B are upper triangular $n \times n$ matrices then AB is upper triangular.

Reason If $A = (a_{ij})$ with $a_{ij} = 0$ if $i > j$
& $B = (b_{ij})$ with $b_{ij} = 0$ if $i > j$

then $AB = C = (c_{ij})$ with
$$c_{ij} = \sum_{k=1}^{n} (a_{ik} b_{kj})$$

We need to show that if $i > j$ then $c_{ij} = 0$.

Now either $k > j$ or $k < i$.

(Either $k > j$ or $k \leq j < i$.)

If $k > j$ then $b_{kj} = 0$ } $\Rightarrow a_{ik} b_{kj} = 0$.
If $k < i$ then $a_{ik} = 0$ } $\Rightarrow c_{ij} = 0$.

Done

Exercise Do the same for strictly lower ~~upper~~ triangular matrices.

2.4 Linear systems in matrix notation

Recall an $n \times 1$ matrix is called a column vector. The set of all such is $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \,\middle|\, x_1, \dots, x_n \in \mathbb{R} \right\}$.

Given an $m \times n$ linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

we can write it in terms of matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & \dots & & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or $A\underset{\sim}{x} = \underset{\sim}{b}$ for short, where $A = (a_{ij})_{m \times n}$

$$\underset{\sim}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\underset{\sim}{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

<u>Example</u>

$$2x_1 - 3x_2 + x_3 = 2$$
$$3x_1 \qquad\quad - x_3 = -1$$

can be written as

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Another way of writing this as

$$x_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Or in the general case we have a <u>vector equation</u>

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

2.5 Next we want to translate <u>elementary row operations</u> into matrix form.

For each row operation write down the matrix obtained by doing this operation to the identity matrix. $(n \times n)$

<u>Type I</u> (swapping two rows)

e.g. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

<u>Type II</u> (multiplying a row by a scalar)

e.g. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

<u>Type III</u> (adding a multiple of one row to another)

e.g. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

These matrices are called <u>elementary matrices</u>.

Example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}+2a_{31} & a_{12}+2a_{32} & a_{13}+2a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

More generally:

Fact Multiplying $E.A$ where E is an <u>elementary $m \times m$ matrix</u>, & A is any $m \times n$ matrix gives the matrix obtained by doing the corresponding <u>elementary row operation</u> to A.

Type I elementary matrix is its own inverse:

e.g. $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Type II also have inverses:

e.g. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\& \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \right)$$

Type III also have inverses:

e.g. $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\times 2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general:

If E is an elementary matrix, it's obtained by applying an elementary row operation to the identity matrix. Hence if F is the matrix of "undoing" that row operation then

$$FE = I.$$

Gaussian elimination is therefore equivalent to multiplying a matrix equation $Ax = b$ on the left by a sequence of elementary matrices:

$$E_1 A x = E_1 b$$
$$E_2 (E_1 (A x)) = E_2 E_1 b$$
$$\vdots$$
$$E_k \cdots (E_1 (A x)) = E_k \cdots (E_1 b)$$

By associativity of matrix multiplication, this is the same as

$$(E_k \cdots E_1) A x = (E_k \cdots E_1) b$$

Similarly, Gauss-Jordan elimination is of this form.

~~Example If A is an invertible matrix~~

~~then what.~~

Terminology 2.30

If $a_1, \dots, a_n$ are vectors in $\mathbb{R}^m$ and $x_1, \dots, x_n$ are scalars, then

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

is a <u>linear combination</u> of $a_1, \dots, a_n$ with <u>weights</u> $x_1, \dots, x_n$.

Fact 2.31 (Consistency Theorem for Linear Systems)

A linear system $Ax = b$ is consistent <u>if and only if</u> b can be written as a linear combination of the columns of A.

Reason

Obvious. $Ax = b$ can be written $x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

Section 2.5 The Invertible Matrix Theorem.

Helpful fact 2.32

Let $Ax = b$ be an $m \times n$ system.

Suppose M is an invertible $m \times m$ matrix.

Then $MAx = Mb$ is an equivalent system to $Ax = b$.

Why? If $Ax = b$ then

$$M(Ax) = Mb$$

$$\text{so } (MA)x = Mb.$$

Conversely, if $MAx = Mb$ and M^{-1} is the inverse of M then $M^{-1}(MAx) = M^{-1}(Mb)$

$$\Rightarrow (M^{-1}M)Ax = (M^{-1}M)b$$

$$\Rightarrow (I_m A)x = I_m b$$

$$\Rightarrow Ax = b$$

Useful consequence

If $Ax = b$ is an $n \times n$ system, and A is invertible, then the system is equivalent to

$$x = A^{-1}b$$

Terminology 2.39

A matrix B is said to be **row equivalent** to A if B can be obtained from A by a sequence of elementary row operations.

Equivalently, if

$$B = E_k E_{k-1} \cdots E_1 A$$

where $E_1, \ldots, E_k$ are elementary matrices.

Fact 2.40

- A is row equivalent to itself
- If B is row equivalent to A, then A is row equivalent to B.
- If B is row equivalent to A, <u>and</u> C is row equivalent to B, then C is row equivalent to A.

Why? • If $B = E_k E_{k-1} \cdots E_1 A$

then $E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1} B =$

$$E_1^{-1} \cdots \cancel{E_k^{-1} E_k} \cdots E_1 A = A$$

• If $B = E_k E_{k-1} \cdots E_1 A$

and $C = F_l F_{l-1} \cdots F_1 \underline{B}$

then $C = \underbrace{F_l \cdots F_1 E_k \cdots E_1}_{\text{elementary matrices.}} A$

Facts 2.4.1 (Invertible Matrix Theorem)

If A is an $n \times n$ matrix, then the following statements are all equivalent:

(a) A is invertible

(b) $Ax = 0$ has only the trivial solution

(c) A is row equivalent to I_n

(d) A is a product of elementary matrices

Proof (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) If A is invertible, & $Ax = 0$, then

$$A^{-1}(Ax) = A^{-1}0 = 0$$

$\Rightarrow x = Ix = A^{-1}Ax = 0.$

<u>(b) ⇒ (c)</u> If $Ax = 0$ has no non-trivial solution, then in the echelon form there are <u>no</u> free variables, and n leading variables.

$$\Rightarrow A \underset{\substack{\uparrow \\ \text{row equivalent to}}}{\sim} \begin{pmatrix} 1 & x & & * \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$ after Gaussian algorithm

$$\Rightarrow A \sim \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$ by Gauss-Jordan.

<u>(c) ⇒ (d)</u> A is row equivalent to I

So $A = \underbrace{E_k E_{k-1} \cdots E_1}_{\text{elementary matrices}} \cdot I$

$= E_k E_{k-1} \cdots E_1.$

(d) $\Rightarrow$ (a) If $A = E_k E_{k-1} \dots E_1$

then $E_1, \dots E_k$ are invertible,

so $E_1^{-1} E_2^{-1} \dots E_k^{-1} A =$

$E_1^{-1} E_2^{-1} \dots \cancel{E_k^{-1}} \cancel{E_k} \dots E_1 = I$

and $A E_1^{-1} E_2^{-1} \dots E_k^{-1} =$

$E_k \dots E_1 E_1^{-1} \dots E_k^{-1} = I.$

$\Rightarrow A^{-1} = E_1^{-1} \dots E_k^{-1}.$

A is invertible and

Done.

Consequence 2.42

If A and C are square matrices and $AC = I$,

then $CA = I$.

Hence $C = A^{-1}$ and $A = C^{-1}$.

Why? $AC = I \Rightarrow C$ is row-equivalent to I

$\Rightarrow C$ is invertible by the Theorem.

$\Rightarrow A = C^{-1}$

$\Rightarrow CA = I$

$\Rightarrow A$ is invertible, and $A^{-1} = C$.

Done.

Section 2.6: Gauss-Jordan inversion

If A is invertible,
then A is row-equivalent to I.
Hence there is a sequence of elementary matrices $E_1, \dots, E_k$ such that

$$E_k E_{k-1} \dots E_1 A = I$$

$$\Rightarrow E_k E_{k-1} \dots E_1 A A^{-1} = I A^{-1}$$

$$\Rightarrow E_k E_{k-1} \dots E_1 I = A^{-1}$$

So the <u>same</u> sequence of elementary row operations converts I into A^{-1}.

Augmented matrix notation

We wrote $(A|b)$ for an $m \times (n+1)$ matrix,
where A is an $m \times n$ matrix
& b is an $m \times 1$ matrix.

More generally if B is an $m \times k$ matrix,
write $(A|B)$ for the obvious $m \times (n+k)$ matrix.

Gauss–Jordan inversion

- Bring the augmented matrix $(A|I)$ to reduced row echelon form.
- If this form is $(I|B)$, then $B = A^{-1}$.
- Otherwise, A is not invertible.

Example 2.43 Show that the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$$

is invertible,

and compute A^{-1}.

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 0 & 3 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 3 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-3R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -1 & 6 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -6 & 3 & -1 \end{array}\right)$$

$$\xrightarrow{R_2-3R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 16 & -8 & 3 \\ 0 & 0 & 1 & -6 & 3 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -31 & 16 & -6 \\ 0 & 1 & 0 & 16 & -8 & 3 \\ 0 & 0 & 1 & -6 & 3 & -1 \end{array}\right)$$

Hence A is invertible, and

$$A^{-1} = \begin{pmatrix} -31 & 16 & -6 \\ 16 & -8 & 3 \\ -6 & 3 & -1 \end{pmatrix}$$

<u>Check</u> $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix} \begin{pmatrix} -31 & 16 & -6 \\ 16 & -8 & 3 \\ -6 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$