A world-wide-web Atlas of group representations
Final Report
GR/S31419

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July 26, 2007

A Background and context

The theory of finite groups is of central importance in mathematics, and finds wide applications in all the physical sciences and elsewhere. In essence it is the deep study of symmetry in all its innumerable manifestations, and so has applications in all situations where symmetry occurs. The building blocks of finite groups are the 'simple' groups, analogous to the prime numbers in number theory, which are so-called because they cannot be broken down into smaller pieces. The study of 'simple' groups, however, just like the study of prime numbers, turns out to be not simple at all.

The completion around 1980 of the massive world-wide project for complete classification of the finite simple groups (see, for example, [31]), revealed that they mostly fall into various reasonably well understood families, *with precisely twenty-six exceptions*, known as the 'sporadic' simple groups. These range in size from the little Mathieu group, known since 1860, which has 7920 elements, to the Fischer–Griess Monster (or Friendly Giant) [32], suspected since 1973 but not constructed until 1980, which has nearly 10^{54} elements.

Now that we know the names of all the finite simple groups, attention has shifted to studying their properties, especially their maximal subgroups, and representation theory (e.g. Brauer character tables). There has been a major worldwide project in calculating the maximal subgroups, representations, characters, cohomology, etc. of the 'generic' simple groups, which is still very much in progress. When it comes to the sporadic groups, however, the 'generic' methods do not apply, and essentially only 'ad hoc' methods are useful. The relevant properties fall into the realm of 'facts' rather than 'theorems', and as such it makes sense to calculate them once and for all, and store the results in a large database.

From the earliest days there have been collections of information about groups, ranging from character tables e.g. of symmetric groups, to complete structural information as required e.g. by crystallographers for point groups and space groups. During the later stages of the classification project, it became clear that huge amounts of information about simple groups needed to be tabulated. This was the motivation for the Cambridge Atlas project which culminated in the publication in 1985 of the 'Atlas of Finite Groups' [29], and later the 'Atlas of Brauer Characters' [33].

More recently, however, a demand has arisen for such data to be available on computers, for input into computational algebra systems. In particular, it is not enough to have the characters of representations, one wants the representations themselves. Up till now, most of the representations available have been over fields of finite characteristic [40], but now the capability exists to provide characteristic zero representations, which is apparently what users outside pure group/representation theory really want.

B Key advances and supporting methodology

B.1 Aims and Objectives

The aim of the project was to further research into and understanding of:

- sporadic (and other) simple groups and related structures, and
- their representations, presentations, cohomology, etc., and
- computational algebra.

The specific objectives of the project were:

1. to compute systematically large numbers of characteristic zero representations of simple and related groups, perhaps up to dimension 100,
2. to obtain presentations for Atlas groups on 'standard' generators (at least up to order 10^9) and other generators,
3. to compute where possible 1-cohomology of simple groups acting on small modules, and $\mathrm{Ext}^1(A, B)$ for small modules A and B,
4. to try to compute 2-cohomology of small groups (though this is hard), and to construct interesting non-split extensions explicitly,
5. to fill gaps in the present Web-Atlas by computing new representations in finite characteristic, and obtaining words in standard generators giving maximal subgroups, automorphisms, conjugacy class representatives, etc.,
6. to obtain small representations of maximal subgroups of sporadic simple groups, especially p-modular representations of p-local subgroups,
7. to provide improved access to the web-atlas, via MAGMA and GAP interfaces, and providing (limited) computational facilities via web-browsers, and
8. to maintain and enhance the current facilities of the Web-Atlas.

B.2 Methodology

Some of the methodology is tried and tested: basic meataxe methods for manipulating representations over finite fields, and constructing new ones, and 'condensation' methods for assisting this process for large representations.

Areas where new methods were developed include the construction of characteristic zero representations. In some cases the amalgam method [38] was used, and in other cases condensation methods were used to extract such representations from permutation representations. Combining this with judicious use of the LLL algorithm often produces integral representations with small matrix entries (sometimes even single digits). Other methods which have been used are rational reconstruction of a representation from its reductions modulo several large primes (this was extended to irrational reconstruction on occasion), and once or twice Minkwitz's method (although it has very limited applicability).

To construct presentations of known groups, we used known representations to find relations, and then used single and double coset enumeration programs (e.g. those developed by G. Havas, S. Linton, and by J. N. Bray) to test possible presentations, typically enumerating cosets of an involution centralizer, found by the methods of [11].

Calculation of 1-cohomology, and more generally $\mathrm{Ext}^1(A, B)$, works by converting the relations of a presentation into linear equations which can be solved by standard Gaussian elimination algorithms. Often if we do not know a presentation, we can still use several relations to get an upper bound on the dimension of the 1-cohomology, and find explicit indecomposable representations to obtain a lower bound.

Construction of non-split group extensions $V \cdot G$ works by gluing together indecomposable representations of G, in such a way that the resulting representation is no longer a representation of G, but of some non-split extension $V \cdot G$. This then gives lower bounds on 2-cohomology. To obtain upper bounds, we had to develop new methods, as current implementations of general purpose algorithms are too slow for the range of problems we wished to attack. For example we used Shapiro's Lemma to reduce to smaller problems of the same kind, and used dimension-shifting to reduce to calculation of 1-cohomology. In the end, the suggestion of applying condensation methods to PIMs (given as summands of suitable permutation modules, for example) turned out to be of limited use.

B.3 Description of the work undertaken

The objectives enumerated above were largely achieved:

1. a systematic calculation of over 650 characteristic zero representations of simple and related groups in dimension up to 250 was carried out by my PhD student Simon Nickerson [36], and incorporated by him into the Web-Atlas. Many others were computed by John Bray, for example a 124-dimensional representation of $Sz(32)$, which was especially challenging. A particularly interesting outcome here was a beautiful new formula found by John Bray [15] which gives a

theoretical limit to how small the entries in the matrices of an integral representation can be. In many instances he was able to find bases which achieve this bound, or prove that none exists.

2. John Bray computed systematically presentations on the standard generators, and other generators, for most of the almost simple groups of order up to 10^9, except some of the groups $L_2(q)$, of which there are many. In particular he found new very short presentations for the alternating and symmetric groups [17]. Other work on (symmetric) presentations begun earlier has continued [10, 18, 19, 20].
3. John Bray undertook a systematic calculation of 1-cohomology, for all simple groups of order up to about 10^7, plus many of the larger sporadic groups, and some of the results have been included in the Web-Atlas. More generally, he calculated many Ext groups, but these are not yet incorporated. Two examples were found [26] of 3-dimensional 1-cohomology of an absolutely irreducible module: no such examples were known before. A new infinite family [14] of modules with 2-dimensional 1-cohomology was also found.
4. Some progress was made on calculating 2-cohomology, mostly for simple groups of order less than 10^6. For example, the 2-cohomologies for all the irreducible modules of the Higman–Sims group in characteristic 2 were calculated. A few bigger examples were calculated, such as the 248-dimensional irreducibles (in the principal block!) for the Thompson group in characteristics 5 and 7. Of particular interest is the 5-dimensional 2-cohomology for a 156-dimensional $GF(3)$-module for $U_4(3)$.
5. The original plan was that the project student would learn the ropes by filling in gaps in the data. When the student was replaced by a PDRA, this seemed inappropriate, so more gaps remain than we would like. One big gap which was filled by Richard Barraclough was a list of conjugacy class representatives in the Monster [9].
6. We wanted to produce small representations of maximal subgroups in order to be able to study them more effectively. For example we made representations of most of the maximal subgroups of the Monster [25] in order not to have to work in a 196882-dimensional representation. These have been used in [2, 27], for example. Similarly for the Baby Monster, we made small representations of the maximal 2-local subgroups, and the rest can be obtained from our representations of subgroups of the Monster. For smaller groups or representations, the impressive computational capabilities of recent versions of MAGMA make our proposed work largely unnecessary.
7. Improved access to the web-atlas via a database-driven interface ('version 3' [40], written largely by Simon Nickerson) has proved to be much appreciated by users. Not all the information has been easy to port from version 2, however, and some work remains to be done. The main part of the data is also available in MAGMA [28], as well as via automatic downloads into GAP [39, 30]. We decided that providing computational facilities via web-browsers was unlikely to be cost-effective, so this part of the objective was abandoned.
8. Maintenance of the system has included large-scale automatic checking of data in the web-atlas. For example, the concept of 'semipresentation' [37] was invented to formalise the process of checking the group generators in cases where no presentation is known or easily available. Much of the day-to-day maintenance was carried out by Richard Barraclough.

C Project plan review

The original project plan incorporated a project studentship, and a student was duly assigned to the project from the beginning. However, when I moved institutions in September 2004, the student wished to stay in Birmingham rather than move to London. At the suggestion of EPSRC, therefore, the funds were diverted to support a second PDRA for a shorter period on the project.

Timing of visits to collaborators and conferences was changed according to availability. Travel money intended for longer visits abroad by the PI, PDRA and the project student was not fully used. Partly this was due to the fact that moving from Birmingham to QMUL removed my entitlement to a sabbatical.

Equipment funds have been re-classified as consumables according to EPSRC rules, since the equipment required was relatively inexpensive.

D Research impact and benefits to society

Mathematicians working with groups greatly appreciate the ability to perform explicit calculations, as the wide use of packages such as GAP and MAGMA testifies. Our world-wide-web atlas has been well used by many group theorists and other mathematicians over the years. Feedback suggests that they much appreciate the improvements that have been made under the auspices of this project. The representations and straight-line-programs that we have collected and calculated are now available directly in GAP [30] and in MAGMA from version 2.11 [28].

For example, Jianbei An (Auckland) is using our constructions of maximal subgroups to verify the Alperin–McKay–Dade–Isaacs–Navarro–Uno conjectures for the Monster ([2], see also [1, 4]). They are also used in [27] to determine minimal base sizes for primitive permutation representations of sporadic groups.

E Explanation of expenditure

Expenditure closely followed the original plan, apart from the changes mentioned above. We paid for (i) a 3-year PDRA (John Bray) to do the majority of the work, (ii) a 1-year PDRA (Richard Barraclough) to assist where necessary, (iii) a short-term appointment (7 weeks) of Simon Nickerson to write the web/database interface, (iv) a computer to host the web-site, (v) a desktop and two laptops for the PDRAs and PI, (vi) travel to conferences etc.

Dr John Bray was appointed at the start of the grant, as envisaged in the application, and moved with me to London after 4 months. Dr Barraclough was appointed in September 2005. He left in September 2006 to take up another job, which meant that the money which had been earmarked for the final few months of his salary remained unspent.

The computer which we bought was a slight upgrade of what was proposed in the application, made possible by continued decreases in the prices of equipment. In addition we were able to buy laptop computers for the PI and RA, and other small items.

Travel money was used to fund (or partly fund) various visits to collaborators: the MAGMA group in Sydney (by Wilson), Roney-Dougal in St Andrews (by Bray), O'Brien and An in Auckland (by Bray and Wilson), the Aachen group (by Bray and Wilson). In addition, we paid for conference attendances in Oberwolfach (Bray, Wilson), Edinburgh Moonshine Workshop (Bray, Wilson), Groups St Andrews 2005 (Barraclough, Bray), Warwick Magma Workshop (Barraclough, Bray), and one- or two-day meetings in Aachen, Oxford, Cambridge, Manchester and Birmingham.

F Further research or dissemination activities

F.1 Other work carried out by the PDRAs

Another significant benefit of the grant was that the two PDRAs were able to contribute to other research projects. In particular, Dr Bray has continued work related to two earlier EPSRC grants (GR/N27491 and GR/R95265). See [10, 18, 19, 20].

He has begun a major research collaboration with Prof. Derek Holt (Warwick) and Dr. Colva Roney-Dougal (St Andrews) aimed at a definitive and explicit classification of all maximal subgroups of all almost-simple finite classical groups up to a reasonable dimension [21].

He has investigated the exceptional aspects of Curtis–Tits–Phan theory for unitary groups where $q = 3$. In particular, the analogue for $q = 3$ of the known presentation for $SU_n(q)$ for $q \geq 4$ is not a presentation for $SU_n(3)$. Indeed, there is a non-split extension of the exterior fourth power of the natural module by $SU_n(3)$, which satisfies this presentation. He is continuing work on the even more exceptional behaviour at $q = 2$. Other work he has done recently includes [2, 3, 5, 22, 23, 24].

Dr Barraclough has continued working on the main project from his PhD [6], namely to classify the 'nets' in the Monster. One constituent of this work was the calculation of the character table of the maximal subgroup $3^{1+12}{\cdot}2Suz{:}2$ [7]. See also [8] for another large character table he needed and therefore calculated.

F.2 Dissemination and exploitation

The main route for dissemination of the results of this project is, and was always intended to be, via the internet. Thus most of our results are made available in the WWW-Atlas [40], although there is still a backlog of results that are not yet included. Much of this material is then accessible from GAP, which will automatically download it from our website when required. Moreover, the bulk of our material is now accessible within MAGMA, either distributed with MAGMA itself, or by downloading MAGMA files from our website.

In addition, where the results and/or methods are of particular interest, they have been and will be disseminated through the usual academic routes of publication in refereed journals and presentation at national and international conferences. There are five publications which have already appeared [7, 9, 25, 37, 40], and ten more submitted or in preparation [2, 3, 12, 13, 14, 15, 16, 17, 26, 27], as well as at least nine publications more indirectly related to the grant [5, 8, 18, 21, 22, 23, 24, 35, 41].

F.3 Further work

There is continuing work on maintaining and updating the web-site. Dr Bray is continuing his work on cohomology, and further results in this area will be added to the web-site when available. However, it is not envisaged that we will continue to spend a great deal of time on upgrading this facility in the future.

Similarly, we shall continue to construct characteristic zero representations, especially for proper covers of simple groups, which are generally more difficult than the simple groups.

We are working on constructing small representations of the remaining (2-local) maximal subgroups of the Monster, which have so far proved resistant to attack.

Richard Barraclough's work on nets in the Monster is intended to be the subject of a future EPSRC grant application. The idea is that we will use a computational version of the Griess algebra to facilitate these and other calculations in the Monster.

John Bray is intending that his work on maximal subgroups of classical groups should develop into similar work on exceptional groups of Lie type, which might also be the subject of a future EPSRC grant application.

References

[1] Jianbei An and R. A. Wilson, The Alperin weight conjecture and Uno's conjecture for the Baby Monster $\mathbb{B}$, p odd, *LMS JCM* **7** (2004), 120–166.

[2] J. An, J. N. Bray and R. A. Wilson, The Alperin weight conjecture and Uno's conjecture for the Monster, $\mathbb{M}$, p odd, in preparation.

[3] J. An, J. N. Bray and R. A. Wilson, The Alperin weight conjecture and Uno's conjecture for the Baby Monster, $\mathbb{B}$, $p = 2$. in preparation.

[4] Jianbei An, E. A. O'Brien and R. A. Wilson, The Alperin weight conjecture and Dade's conjecture for the simple group J_4, *LMS JCM* **6** (2003), 119–140.

[5] R. F. Bailey and J. N. Bray, Decoding the Mathieu group M_{12}, submitted.

[6] R. W. Barraclough, *Some calculations related to the Monster group*, PhD thesis, University of Birmingham, 2005. (available from `http://www.maths.qmul.ac.uk/%7Eraw/`)

[7] R. W. Barraclough, The character table of a maximal subgroup of the Monster, *LMS JCM* **10** (2007), 161–175.

[8] R. W. Barraclough, The character table of a group of shape $(2 \times 2{}^{\cdot}G){:}2$, preprint, 2006.

[9] R. W. Barraclough and R. A. Wilson, Conjugacy class representatives in the Monster group, *LMS JCM* **8** (2005), 205–216.

[10] S. W. Bolt, J. N. Bray, R. T. Curtis, Symmetric generation of the Janko group J_4, submitted.

[11] J. N. Bray, An improved method for generating the centralizer of an involution, *Arch. Math. (Basel)* (2000), **74**, 241–245.

[12] J. N. Bray, Constructing minimal matrix representations of non-split extensions of $SL_n(q)$ by the natural module, in preparation.

[13] J. N. Bray, Faithful permutation representations of minimal degree of non-split groups $\mathbb{F}_q^n{}^{\cdot}L_n(q)$, in preparation.

[14] J. N. Bray, A new family of modules with 2-dimensional 1-cohomology, submitted.

[15] J. N. Bray, Good bases for lattices, in preparation.

[16] J. N. Bray, On Curtis–Tits–Phan theory for $SU_n(3)$, in preparation.

[17] J. N. Bray, M. Conder, C. R. Leedham-Green, E. A. O'Brien, Short presentations for alternating and symmetric groups, submitted.

[18] J. N. Bray and R. T. Curtis, The Leech lattice Λ and the Conway group $\cdot 0$ revisited, submitted.

[19] J. N. Bray and R. T. Curtis, Double coset enumeration of symmetrically generated groups, *J. Group theory* **7** (2004), 167–185.

[20] J. N. Bray, R. T. Curtis, C. W. Parker, C. B. Wiedorn, Symmetric presentations for the Fischer groups II: the sporadic groups. *Geom. Dedicata*, **112** (2005), 1–23.

[21] J. N. Bray, D. F. Holt and C. Roney-Dougal, *Maximal subgroups of finite classical groups*, book in preparation.

[22] J. N. Bray, R. A. Wilson and J. S. Wilson, A characterization of finite soluble groups by laws in two variables, *Bull. London Math. Soc.* **37** (2005), 179–186.

[23] J. N. Bray and R. A. Wilson, On the orders of automorphism groups of finite groups, *Bull. London Math. Soc.* **37** (2005), 381–385.

[24] J. N. Bray and R. A. Wilson, On the orders of automorphism groups of finite groups. II, *J. Group Theory* **9** (2006), 537–545.

[25] J. N. Bray and R. A. Wilson, Explicit representations of maximal subgroups of the Monster, *J. Algebra* **300** (2006), 834–857.

[26] J.N. Bray and R. A. Wilson, Examples of 3-dimensional 1-cohomology for absolutely irreducible representations of finite simple groups, submitted.

[27] T.C. Burness, E.A. O'Brien and R. A. Wilson, Base sizes for sporadic simple groups, in preparation.

[28] J. Cannon et al., MAGMA, University of Sydney.

[29] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups* (Oxford University Press, Oxford, 1985).

[30] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4.9*; 2006, (`http://www.gap-system.org`).

[31] D. Gorenstein, R. Lyons, R. Solomon, *The classification of the finite simple groups*, vol. 40 of AMS Mathematical surveys and monographs, AMS, Providence, RI, 1994.

[32] R. L. Griess, The Friendly Giant, *Invent. Math.* **69** 1–102 (1982).

[33] C. Jansen, K. Lux, R. A. Parker, R. A. Wilson, *ATLAS of Brauer Characters*, LMS Monographs 11, Oxford University Press, Oxford, 1995.

[34] F. Lübeck and M. Neunhöffer, Enumerating large orbits and direct condensation, *Experiment. Math.* **10** (2001), 197–205.

[35] J. Müller, M. Neunhöffer and R. A. Wilson, Enumerating big orbits and an application: B acting on the cosets of Fi_{23}, *J. Algebra*, **314** (2007), 75-96.

[36] Simon Nickerson, An Atlas of characteristic zero representations, Ph D thesis, University of Birmingham, 2005. `http://web.mat.bham.ac.uk/S.Nickerson/`

[37] S. J. Nickerson and R. A. Wilson, Semi-presentations for the sporadic simple groups, *Exp. Math.* **14** (2005), 359–371.

[38] R. A. Parker, R. A. Wilson, *The Computer Construction of Matrix Representations of Finite Groups over Finite Fields*, J. Symbolic Computation (1990) **9**, 583–590.

[39] R. A. Wilson, R. A. Parker, J. N. Bray and T. Breuer, AtlasRep—A world-wide-web Atlas of Group Representations, `http://www.mat.bham.ac.uk/atlas/` *published as a (refereed) on-line share package for the GAP4 system*, 2001.

[40] R. A. Wilson, S. J. Nickerson and J. N. Bray, *Atlas of Finite Group Representations*, `http://brauer.maths.qmul.ac.uk/Atlas/v3/`, 2005/6/7

[41] R. A. Wilson, An elementary construction of the Ree groups of type, ${}^2G_2$, submitted.