Finite simple groups **Problem sheet 2**

EXERCISE 1. *Compute the addition and multiplication tables for the fields*

1. $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)$;
2. $\mathbb{F}_8 = \mathbb{F}_2[x]/(x^3 + x + 1)$;
3. $\mathbb{F}_9 = \mathbb{F}_3[x]/(x^2 + 1)$.

EXERCISE 2. *Let* $G = \mathrm{GL}_n(q)$. *Prove that* $Z(G) = \{\lambda I_n \mid 0 \neq \lambda \in \mathbb{F}_q\}$, *where* I_n *is the* $n \times n$ *identity matrix.*

EXERCISE 3. *Prove that* $\mathrm{GL}_2(2) \cong S_3$.

EXERCISE 4. *Prove that* $\mathrm{PGL}_2(3) \cong S_4$ *and* $\mathrm{PSL}_2(3) \cong A_4$.

EXERCISE 5. *Prove the following generalisation of Iwasawa'a Lemma: Suppose that* G *is a finite perfect group acting faithfully and primitively on a set* Ω, *and suppose that the stabilizer of a point has a normal soluble subgroup* S, *whose conjugates generate* G. *Then* G *is simple.*

EXERCISE 6. *Use Iwasawa's Lemma to prove simplicity of the alternating groups* A_n *for* $n \geq 5$. *Where does your proof break down for* $n \leq 4$?

[Hint: use the action of A_n *on unordered triples from* $\{1, 2, \ldots, n\}$*.]*

EXERCISE 7. *Let* A *be a* $k \times k$ *matrix,* B *a* $k \times m$ *matrix and* C *an* $m \times m$ *matrix over a field* F. *Show that* $\det \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} = \det A . \det C$.

Deduce that the group G *of invertible matrices of the shape* $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ *has a normal subgroup* $U = \left\{ \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \right\} \cong F^{km}$ *and a subgroup* $L = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \right\} \cong \mathrm{GL}_k(F) \times \mathrm{GL}_m(F)$.

Show also that this group is a semidirect product of U *and* L.

EXERCISE 8. *How many* k*-dimensional subspaces are there in a vector space of dimension* n *over the field of* q *elements?*

EXERCISE 9. *Prove that the stabilizer in* $\mathrm{GL}_n(q)$ *of a* k*-dimensional subspace of* ${\mathbb{F}_q}^n$ *is a maximal subgroup of* $\mathrm{GL}_n(q)$.

EXERCISE 10. *Show that in* $V = \mathbb{F}_4^3$ *there are* 21 *1-spaces ('points') and* 21 *2-spaces ('lines'), that every line contains* 5 *points and that every point lies in* 5 *lines.*

EXERCISE 11. *Prove the equivalence of the following alternative definitions of the Frattini subgroup* $\Phi(G)$.

- $\Phi(G)$ *is the intersection of all the maximal subgroups of* G.
- $\Phi(G)$ *is the set of non-generators of* G, *where* $x \in G$ *is a* non-generator *if for all subsets* $X \subseteq G$, $\langle X \rangle = G \Rightarrow \langle X \setminus \{x\} \rangle = G$.

EXERCISE 12. *Prove that* $\Phi(G)$ *is nilpotent.*

EXERCISE 13. *Prove that if* G *is nilpotent then* $\Phi(G) \geq G'$.