Lecture 5: Sporadic simple groups

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INTRODUCTION

Sporadic simple groups

The 26 sporadic simple groups may be roughly divided into five types:

- The five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}:
  —permutation groups on 11, ..., 24 points.
- The seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2:
  —(real) matrix groups in dimension at most 24.
- The three Fischer groups Fi_{22}, Fi_{23}, Fi'_{24}:
  —automorphism groups of rank 3 graphs.
- The five Monstrous groups $\mathbb{M}$, $\mathbb{B}$, Th, HN, He:
  —centralisers in the Monster of elements of order 1, 2, 3, 5, 7.
- The six pariahs J_1, J_3, J_4, ON, Ly, Ru:
  —oddments which have little to do with each other.

MATHIEU GROUPS

The hexacode

- $F_4 = \{0, 1, \omega, \overline{\omega}\}$ is the field of order 4
- Take six coordinates, grouped into three pairs
- Let the hexacode $\mathcal{C}$ be the 3-space spanned by

$$\begin{array}{c|c|c} (\ \omega \ \ \overline{\omega} & \overline{\omega} \ \ \omega & \overline{\omega} \ \ \omega\ ) \\ (\ \overline{\omega} \ \ \omega & \omega \ \ \overline{\omega} & \overline{\omega} \ \ \omega\ ) \\ (\ \overline{\omega} \ \ \omega & \overline{\omega} \ \ \omega & \omega \ \ \overline{\omega}\ ) \end{array}$$

- This is invariant under
  - scalar multiplications,
  - permuting the three pairs, and
  - reversing two of the three pairs.

The hexacode, II

- This group $3 \times S_4$ has four orbits on non-zero vectors in the code:
  - 6 of shape $(11|\omega\omega|\overline{\omega}\overline{\omega})$
  - 9 of shape $(11|11|00)$
  - 12 of shape $(\omega\overline{\omega}|\omega\overline{\omega}|\omega\overline{\omega})$
  - 36 of shape $(01|01|\omega\overline{\omega})$
- The full automorphism group of this code is $3A_6$, obtained by adjoining the map

$$(ab|cd|ef) \mapsto (\omega a, \overline{\omega} b|cf|de)$$

- We can extend to $3S_6$ by mapping

$$(ab|cd|ef) \mapsto (\overline{a}\overline{b}|\overline{c}\overline{d}|\overline{f}\overline{e})$$

The binary Golay code

- Take 24 coordinates (for a vector space over F_2), corresponding to $0, 1, \omega, \overline{\omega}$ in each of the six coordinates of the hexacode:

0	•	•	•	•	•	•
1	•	•	•	•	•	•
ω	•	•	•	•	•	•
$\overline{\omega}$	•	•	•	•	•	•

- For each column, add up the (entry 0 or 1)$\times$(row label 0, 1, ω or $\overline{\omega}$)
- These six sums must form a hexacode word.
- The parity of each column equals the parity of the top row (even or odd).

Some Golay code words

0	1					
1		1	1	1		
ω			1		1	
$\overline{\omega}$			1			1
	0	1	0	1	ω	$\overline{\omega}$

1	1				
1	1			1	1
1			1		1
1			1	1	
0	1	0	1	ω	$\overline{\omega}$

1	1	1		1	
1	1	1			1
1		1	1	1	1
1		1	1		
0	1	0	1	ω	$\overline{\omega}$

The weight distribution

- For each of the 64 hexacode words, there are 2^5 even and 2^5 odd Golay codewords, making $2^{12} = 4096$ altogether.
- The 32 odd words are 6 of weight 8, 20 of weight 12 and 6 of weight 16.
- The 32 even words are
  - For the hexacode word 000000, one word each of weight 0 or 24, and 15 words each of weight 8 or 16
  - For each of the 45 hexacode words of weight 4, 8 Golay code words of each weight 8 or 16, and 16 of weight 12.
  - For each of the 18 hexacode words of weight 6, we get all 32 Golay code words of weight 12.
- So the full weight distribution is $0^1 8^{759} 12^{2576} 16^{759} 24^1$.

The cosets of the code

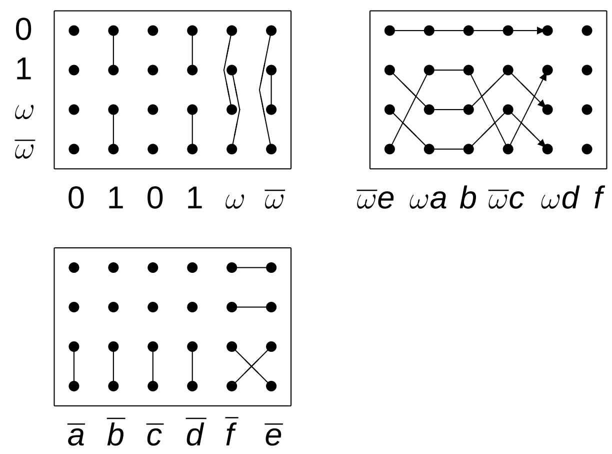
- The Golay code is a vector subspace of dimension 12 in ${F_2}^{24}$, so has $2^{12} = 4096$ cosets.
- The sum (= difference) of two representatives of the same coset is in the code, so has weight at least 8.
- The identity coset has representative $(0^{24}) = (000000000000000000000000)$
- There are 24 cosets with representative of shape $(1, 0^{23})$.
- There are $24.23/2 = 276$ cosets with representatives of shape $(1^2, 0^{22})$.
- There are $24.23.22/3.2 = 2024$ cosets with representatives $(1^3, 0^{21})$.

The sextets

- This leaves exactly $1771 = 24.23.22.21/4.3.2.6$ more cosets, so each one has six representatives of shape $(1^4, 0^{20})$
- Each of these 1771 cosets defines a sextet, that is a partition of the 24 coordinates into 6 sets of 4.

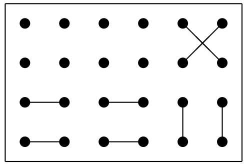
The group $2^6 3S_6$

The 2^6 consists of 'adding a hexacode word' to the labels on the rows.
The $3S_6$ arises from automorphisms of the hexacode.



Generating M_{24}

- M_{24} may be defined as the automorphism group of the Golay code, that is the set of permutations of 24 points which preserve the set of codewords.
- Besides the subgroup $2^6 3S_6$, we can show that the permutation



  preserves the code.
- We can show that M_{24} acts transitively on the 1771 sextets, and that the stabilizer of a sextet is $2^6 3S_6$
- Therefore $|M_{24}| = 244823040 = 2^{10}.3^3.5.7.11.23$.

Properties of M_{24}

- M_{24} acts 5-transitively on the 24 points:
  - Every set of 4 points is a part of a sextet.
  - The sextet stabilizer is transitive on the 6 parts (tetrads), and a full S_4 of permutations of one part can be achieved.
  - Fixing the four points in the first part is a group $2^4 A_5$ which is transitive on the remaining 20 points.
- M_{24} is simple: can be proved using Iwasawa's Lemma applied to the primitive action on the sextets.

Subgroups of M_{24}, I

- The stabilizer of a point has index 24, and is called M_{23}. It has order 10200960.
- The stabilizer of two points is M_{22}, and has order 443520.
- The stabilizer of three points has order 20160, and is in fact $PSL_3(4)$, acting 2-transitively on the $4^2 + 4 + 1 = (4^3 - 1)/(4 - 1)$ points of the projective plane of order 4.

Subgroups of M_{24}, II

- The stabilizer of an octad (a codeword of weight 8) has the shape $2^4 A_8$, acting as A_8 on the 8 coordinates, and as $AGL_4(2)$ on the remaining 16.
- The stabilizer of a dodecad (a codeword of weight 12) has order $|M_{24}|/2576 = 95040$ and is called M_{12}. It acts 5-transitively on the 12 points in the dodecad.
- The stabilizer of a point in M_{12} is called M_{11} and has order 7920.
- The five groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24} are called the Mathieu groups, after Mathieu who discovered them in the 1860s and 1870s.

COFFEE BREAK

THE LEECH LATTICE AND THE CONWAY GROUPS

The Leech lattice

Consider the set of integer vectors $(x_1, \ldots, x_{24}) \in \mathbb{Z}^{24}$ which satisfy

- $x_i \equiv m \bmod 2$, that is either all the coordinates are even, or they are all odd;
- $\sum_{i=1}^{24} x_i \equiv 4m \bmod 8$, and
- for each k, the set $\{i \mid x_i \equiv k \bmod 4\}$ is in the Golay code.

This set of vectors, with the inner product $(x_i).(y_i) = \frac{1}{8}\sum_{i=1}^{24} x_i y_i$, is called the Leech lattice.

The group $2^{12}M_{24}$

- The lattice is clearly invariant under the group M_{24} acting by permuting the coordinates.
- It is also invariant under changing sign on the coordinates corresponding to a Golay code word.
- These sign changes form an elementary abelian group of order 2^{12}, that is a direct product of 12 copies of C_2. This is normalised by M_{24}.
- Together these generate a group of shape $2^{12}M_{24}$ of order $2^{12}|M_{24}| = 4096.244823040 = 1002795171840$.

The minimal vectors

- If the coordinates are odd, assume that all coordinates are $\equiv 1 \bmod 4$.
- Since $24 \equiv 0 \bmod 8$, the smallest norm is achieved when the vector has shape $(-3, 1^{23})$. The norm is then $\frac{1}{8}(9+23) = 4$.
- If the coordinates are even, and some are 2 mod 4, then at least 8 of them are 2 mod 4, and the minimal norm is achieved with vectors of shape $(2^8, 0^{16})$.
- Otherwise they are all divisible by 4, and the minimal norm is achieved with vectors of shape $(4^2, 0^{22})$.

The minimal vectors, II

- Under the action of the group $2^{12}M_{24}$, there are $24.2^{12} = 98304$ images of the vector $(-3, 1^{23})$.
- For each of the 759 octads, there are 2^7 vectors of shape $(\pm 2^8, 0^{16})$, making $2^7.759 = 97152$ in all.
- There are $(24.23/2).2^2 = 1104$ vectors of shape $(\pm 4^2, 0^{22})$.
- So in total there are $98304 + 97152 + 1104 = 196560$ vectors of norm 4 in the Leech lattice.

Vectors of norms 6 and 8

- Similar arguments can be used to classify the vectors of norms 6 and 8.
- The 16773120 vectors of norm 6 are
  - $2^{11}.2576$ of shape $(\pm 2^{12}, 0^{12})$
  - 24.2^{12} of shape $(5, 1^{23})$
  - $(24.23.22/3.2).2^{12}$ of shape $(-3^3, 1^{21})$
  - $759.16.2^8$ of shape $(2^7, -2, 4)$
- There are 398034000 vectors of norm 8.

Congruence classes modulo 2

- Define two vectors in the Leech lattice to be congruent mod 2 if their difference is twice a Leech lattice vector.
- Every vector is congruent to its negative.
- If x is congruent to y, we may assume $x.y$ is positive so that $16 \leq (x-y).(x-y) \leq x.x + y.y$
- Therefore the only non-trivial congruences between vectors of norm ≤ 8 are between perpendicular vectors of norm 8.
- This accounts for at least

$$1 + 196560/2 + 16773120/2 + 398034000/48$$

  congruence classes.
- But this $= 2^{24}$, so these are all.

Crosses

- In particular, each congruence class of norm 8 vectors consists of 24 pairs of mutually orthogonal vectors, called a cross.
- Thus there are 8292375 crosses.
- The stabilizer of a cross (or coordinate frame) is $2^{12}M_{24}$
- By verifying that the linear map which acts on each column as

$$\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

preserves the lattice, we can show that the automorphism group of the lattice acts transitively on the crosses.

Conway's group

- Hence this automorphism group has order 8315553613086720000.
- It has a centre of order 2, consisting of the scalars ± 1.
- Quotienting out the centre gives a simple group Co_1 Conway's first group of order 4157776806543360000.
- Using the primitive action on the 8292375 crosses, and Iwasawa's Lemma, we can easily prove that Co_1 is simple.

Subgroups of the Conway group

- The automorphism group is transitive on the norm 4 vectors, and the stabiliser of a norm 4 vector is Co_2, Conway's second group
- Similarly, the stabiliser of a norm 6 vector is Co_3, Conway's third group
- The stabiliser of two norm 4 vectors whose sum has norm 6 is the McLaughlin group *McL*
- The stabiliser of two norm 6 vectors whose sum has norm 4 is the Higman–Sims group *HS*

THE END