Lecture 4: Exceptional groups of Lie type

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LTCC, 27th October 2008

INTRODUCTION

Exceptional groups

The families we are looking at today are

- $G_2(q)$
- $F_4(q)$
- $E_6(q)$
- $E_7(q)$
- $E_8(q)$
- ${}^2E_6(q)$
- ${}^3D_4(q)$
- ${}^2B_2(q)$
- ${}^2G_2(q)$
- ${}^2F_4(q)$

Lie algebras

The ten families of exceptional finite simple groups of Lie type are all derived in some way from Lie algebras.
A Lie algebra is a vector space with a product, usually written $[x, y]$, satisfying $[y, x] = -[x, y]$ and

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

the Jacobi identity.
The canonical example is the vector space of $n \times n$ matrices of trace 0, with $[x, y] = xy - yx$. This is called $\mathfrak{sl}_n$, and corresponds to the group SL_n of matrices of determinant 1.
Similarly we can make Lie algebras corresponding to the symplectic and orthogonal groups.

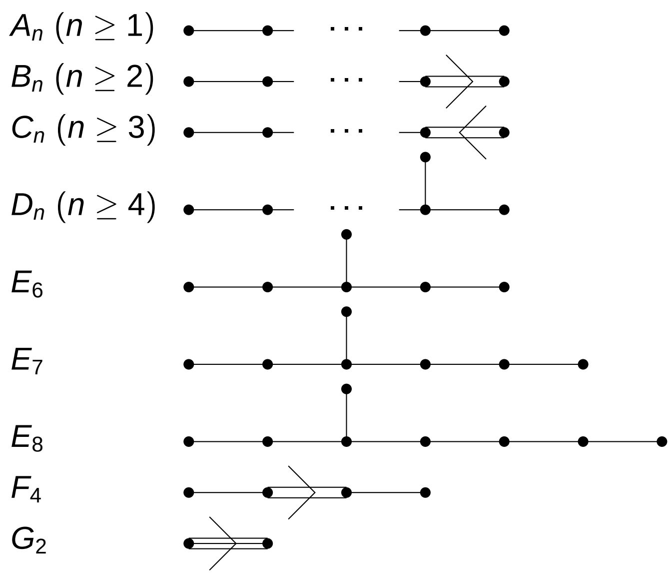
Simple Lie algebras

The simple Lie algebras over $\mathbb{C}$ are

- A_n, also known as $\mathfrak{sl}_{n+1}$
- B_n, also known as $\mathfrak{so}_{2n+1}$
- C_n, also known as $\mathfrak{sp}_{2n}$
- D_n, also known as $\mathfrak{so}_{2n}$
- Five exceptional algebras, G_2, F_4, E_6, E_7, and E_8.

G_2 is the algebra of derivations of the octonion algebra (Cayley numbers)
F_4 is the algebra of derivations of the exceptional Jordan algebra

Coxeter–Dynkin diagrams



Chevalley groups

Corresponding to the exceptional Lie algebras are some exceptional groups of Lie type:

- $G_2(q)$ constructed by L. E. Dickson around 1901, fixing a cubic form on a 7-space.
- $E_6(q)$ constructed by L. E. Dickson around 1905, fixing a cubic form on a 27-space.
- $F_4(q)$ constructed by C. Chevalley around 1955, acting on a Lie algebra of dimension 52
- $E_7(q)$, acting on a Lie algebra of dimension 133
- $E_8(q)$, acting on a Lie algebra of dimension 248.

Twisted groups

- The A_n diagram has an automorphism reversing the order of the nodes. This gives rise to the unitary groups by a kind of twisting operation.
- The D_n diagram has an automorphism swapping the two branches of length 1. This gives rise to the orthogonal groups of minus type.
- The E_6 diagram has an automorphism, giving rise to groups called ${}^2E_6(q)$.
- The D_4 diagram has an automorphism of order 3, giving rise to groups called ${}^3D_4(q)$.

The Suzuki and Ree groups

Some of the diagrams have automorphisms only if we ignore the arrows. For reasons we won't go into, this makes sense only if the characteristic of the field is equal to the multiplicity of the edge.

- The Suzuki groups $Sz(2^{2n+1}) = {}^2B_2(2^{2n+1})$
- The small Ree groups $R(3^{2n+1}) = {}^2G_2(3^{2n+1})$
- The large Ree groups $R(2^{2n+1}) = {}^2F_4(2^{2n+1})$

These turn out to be simple if and only if $n \geq 1$.
$Sz(2)$ has order 20.
$R(3)' \cong PSL_2(8)$ has index 3 in $R(3)$.
$R(2)'$ has index 2 in $R(2)$ and is a simple group not appearing elsewhere in the classification.

OCTONIONS AND G_2

Quaternions

Hamilton's quaternions

$$\mathbb{H} = \mathbb{R}[i, j, k]$$

where

- $i^2 = j^2 = k^2 = -1$
- $ij = k = -ji, jk = i = -kj, ki = j = -ik$

It has an involution

$$^{-} : a + bi + cj + dk \mapsto a - bi - cj - dk$$

called quaternion conjugation, and a norm $N(q) = q\overline{q}$ which satisfies $N(xy) = N(x)N(y)$.

Octonions

- $\mathbb{O} = \mathbb{R}[i_0, i_1, \ldots, i_6]$ with subscripts read modulo 7
- i_t, i_{t+1}, i_{t+3} multiply like i, j, k in the quaternions
- $(i_0 i_1) i_2 = i_3 i_2 = -i_5$
- $i_0(i_1 i_2) = i_0 i_4 = i_5$
- So $\mathbb{O}$ is non-associative.
- It still has an involution

$$^{-} : a + \sum_{t=0}^{6} b_t i_t \mapsto a - \sum_{t=0}^{6} b_t i_t$$

- and a norm $N(x) = x\overline{x}$ which satisfies $N(xy) = N(x)N(y)$.

$G_2(q)$, for q odd

- A corresponding octonion algebra exists with coefficients in any finite field of odd characteristic.
- $G_2(q)$ is the group of linear maps which preserve the norm and the multiplication.
- In particular, it is inside the orthogonal group $O_8^+(q)$, and fixes 1, so is inside $O_7(q)$.
- By counting generating sets equivalent to (i_0, i_1, i_2) we can show that

$$|G_2(q)| = q^6(q^6 - 1)(q^2 - 1).$$

- The contruction is more complicated in characteristic 2.
- $G_2(q)$ is simple for all $q > 2$.
- $G_2(2) \cong PSU_3(3).2$, where the automorphism is the field automorphism of F_9.

EXCEPTIONAL JORDAN ALGEBRAS AND F_4

The Moufang identity

- The octonions satisfy the Moufang identity

$$(xy)(zx) = (x(yz))x$$

which substitutes for the associative law in some ways.
- In particular, if $xyz = 1$ and u satisfies $u\overline{u} = 1$ then

$$\begin{aligned} ((ux)(yu))(\overline{u}z\overline{u}) &= (u(xy)u)(\overline{u}z\overline{u}) \\ &= u(xy)u\overline{u}z\overline{u} \\ &= u(xy)z\overline{u} = 1 \end{aligned}$$

- Therefore the triple of maps

$$(L_u, B_u, R_u) : (x, y, z) \mapsto (ux, yu, \overline{u}z\overline{u})$$

preserves the property that $xyz = 1$.

Triality

- Such a triple (α, β, γ) of maps is called an isotopy.
- There are exactly two isotopies for each $\alpha \in \Omega_8^+(q)$, so the isotopies generate a double cover of the orthogonal group, called the spin group.
- If (α, β, γ) is an isotopy, then (β, γ, α) is an isotopy.
- This is know as the triality automorphism of $P\Omega_8^+(q)$.
- The centralizer of the triality automorphism is the set of isotopies of the form (α, α, α).
- This is none other the automorphism group of the octonions, that is $G_2(q)$.

Jordan algebras

- Jordan algebras were introduced to axiomatise the matrix product $A \circ B = (\frac{1}{2})(AB + BA)$, in an attempt to find a suitable model for quantum mechanics.
- The essential axiom is the Jordan identity

$$((A \circ A) \circ B) \circ A = (A \circ A) \circ (B \circ A).$$

- It turned out that there was only one new algebra, of dimension 27.
- So it was useless for quantum mechanics, but very interesting for group theory.

The exceptional Jordan algebra

- Take 3×3 Hermitian matrices over the octonions, that is matrices of the form

$$\begin{pmatrix} a & C & \overline{B} \\ \overline{C} & b & A \\ B & \overline{A} & c \end{pmatrix}$$

  where a, b, c are real and A, B, C are octonions.
- The Jordan product of two such matrices is still Hermitian.
- There is a corresponding exceptional Jordan algebra with coefficients in any field of characteristic not 2 or 3.
- The construction is more complicated in characteristics 2 and 3.

$F_4(q)$ in characteristic not 2 or 3

- $F_4(q)$ is defined as the automorphism group of the exceptional Jordan algebra over F_q.
- To calculate its order, we count the primitive idempotents, which are defined as elements e with $e \circ e = e$ and having trace 1.
- There are $q^8(q^8 + q^4 + 1)$ of them, and
- the stabiliser of one of them is a double cover of $SO_9(q)$.
- Hence

$$|F_4(q)| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1).$$

$E_6(q)$ in characteristic not 2 or 3

- Surpisingly, the determinant of the Hermitian octonion matrices makes sense!
- $$\det \begin{pmatrix} a & C & \overline{B} \\ \overline{C} & b & A \\ B & \overline{A} & c \end{pmatrix} = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + \Re(ABC) + \Re(CBA)$$
- The group of linear maps which preserve this cubic form is (modulo scalars) $E_6(q)$.

The order of $E_6(q)$

- The notion of rank of Hermitian octonion matrices also makes sense, though needs care to define.
- It can be shown that $E_6(q)$ acts transitively on the matrices of determinant 1.
- One of these is the identity matrix, whose stabilizer is $F_4(q)$. Why?
- Hence we get the order of $E_6(q)$ (modulo scalars of order $(3, q-1)$):

$$q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)/(3, q-1).$$

THE END