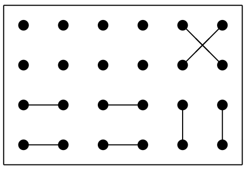
Finite simple groups **Problem sheet 5**

Exercise 1. *Show that the permutation*



preserves the Golay code, and fuses the four orbits of the sextet group $2^6{:}3{}^{\cdot}S_6$ *on sextets into a single orbit.*

Exercise 2. *Prove that the stabiliser of a sextet in the automorphism group of the extended binary Golay code is exactly* $2^6{:}3{}^{\cdot}S_6$ *(and no larger).*

Exercise 3. *Show that* M_{24} *acts transitively on the set of* 2576 *dodecads in the extended binary Golay code.*

Exercise 4. *Construct a 'Leech triangle' for the number of dodecads meeting* $\{1,\ldots,j-1\}$ *in* $\{1,\ldots,i-1\}$*, where* $\{1,\ldots,8\}$ *is an octad of the Steiner system* $S(5,8,24)$.

Exercise 5. *Using the following numbering of the MOG,*

0	∞	1	11	2	22
19	3	20	4	10	18
15	6	14	16	17	8
5	9	21	13	7	12

(1)

show that for each k*, the set* $\{x^2+k \mid x \in \mathbb{F}_{23}\}$ *is a dodecad. Deduce that* $\mathrm{PSL}_2(23)$ *is a subgroup of* M_{24}.

Exercise 6. *Prove simplicity of* M_{24} *using Iwasawa's Lemma applied to the permutation action on the* 759 *octads in the extended binary Golay code.*

Exercise 7. *Prove that* $\mathrm{M}_{21} \cong \mathrm{PSL}_3(4)$.

Exercise 8. *Prove that* M_{22} *is simple by applying Iwasawa's Lemma to the action on the* 77 *hexads.*

Exercise 9. *Prove that* M_{23} *is simple by applying Iwasawa's Lemma to the action on the* 253 *heptads.*

Exercise 10. *Classify the orbits of* $2^{12}{:}\mathrm{M}_{24}$ *on the vectors of norm* 8 *in the Leech lattice, and on the crosses. Deduce that* Co_1 *acts primitively on the set of* 8292375 *crosses.*

EXERCISE 11. *Prove that* $2^{\cdot}\mathrm{Co}_1$ *is transitive on the vectors of norm* 4 *in the Leech lattice, and on the vectors of norm* 6.

EXERCISE 12. *Apply Iwasawa's Lemma to the subgroup* $2^{10}{:}\mathrm{M}_{22}{:}2$ *of* Co_2 *to prove that* Co_2 *is simple.*

EXERCISE 13. *Let* Γ *be the graph on the* 100 *Leech lattice vectors of norm* 4 *with inner product* 3 *with each of the vectors* $(1,5,1^{22})$ *and* $(5,1,1^{22})$*, defined by joining two vectors if and only if their inner product is* 1. *Show that* Γ *is isomorphic to the Higman–Sims graph* Δ *with vertex set* $\{*\}\cup S\cup H$*, where* S *is a* 22*-element set,* H *is the set of* 77 *hexads of a Steiner system* $S(3,6,22)$ *on* S*, and* $s\in S$ *is joined to* $*$ *and the hexads not containing* s*, and two hexads are joined in* Δ *if and only if they are disjoint as subsets of* S.