Lecture 3: Classical groups

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INTRODUCTION

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  - $P\Omega^{+}_{2n}(q)$;
  - $P\Omega^{-}_{2n}(q)$.

Bilinear forms

A bilinear form on a vector space V is a map $B : V \times V \to F$ satisfying

$$\begin{aligned} B(\lambda u + v, w) &= \lambda B(u, w) + B(v, w), \\ B(u, \lambda v + w) &= \lambda B(u, v) + B(u, w) \end{aligned}$$

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An alternating bilinear form is always skew-symmetric, but the converse is true if and only if the characteristic is not 2. Why?

Quadratic forms

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In characteristic 2, the associated bilinear form is alternating, since

$$0 = Q(v + v) = 2Q(v) + B(v, v) = B(v, v).$$

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- $B(w, v) = \overline{B(v, w)}$.
- Consequently $B(u, \lambda v + w) = \overline{\lambda} B(u, v) + B(u, w)$.

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- B is non-singular if $\mathrm{rad}(B) = 0$, and singular otherwise.
- Similarly the radical of Q is the subspace of isotropic vectors in the radical of the associated B.
- A subspace is non-singular if the form restricted to the subspace is non-singular.
- A subspace is totally isotropic if the form restricted to the subspace is identically zero.

Isometries and similarities

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$$B(u^\phi, v^\phi) = \lambda_\phi B(u, v)$$

or

$$Q(v^\phi) = \lambda_\phi Q(v).$$

Classification of alternating bilinear forms

If we can find vectors u, v such that $B(u, v) = \lambda \neq 0$, then take our first two basis vectors to be u and $\lambda^{-1}v$, so that the form has matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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When there are no such vectors left, the form is identically zero.
Notice that the rank of B is always even.
Up to change of basis, there is a unique non-singular form.

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Now we can replace two copies of α by two copies of 1, by picking λ and μ such that $\lambda^2 + \mu^2 = \alpha^{-1}$, and changing basis via $x' = \lambda x + \mu y$ and $y' = \mu x - \lambda y$.

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In this case there are exactly two non-singular forms, up to change of basis.

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The first one has matrix equal to the identity matrix, and is called of plus type.
The second one has a 2×2 block $\begin{pmatrix} 1 & 1 \\ 0 & \mu \end{pmatrix}$ where $x^2 + x + \mu$ is irreducible over F_q, and is called of minus type.

Witt's Lemma

If (V, B) and (W, C) are isometric spaces, with B and C non-singular, and either

- alternating bilinear, or
- conjugate-symmetric sesquilinear, or
- symmetric bilinear in odd characteristic

then any isometry between a subspace X of V and a subspace Y of W extends to an isometry of V with W.

COFFEE BREAK

DEFINITIONS OF THE CLASSICAL GROUPS

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Of the $q^{2n} - q$ vectors which are linearly independent of the first, $q^{2n-1} - q$ are orthogonal to it, and q^{2n-1} have each non-zero inner product. So there are q^{2n-1} choices for the second vector.

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By induction on n, the order of $Sp_{2n}(q)$ is

$$\prod_{i=1}^{n}(q^{2i} - 1)q^{2i-1} = q^{n^2}\prod_{i=1}^{n}(q^{2i} - 1).$$

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- $Sp_4(2) \cong S_6$.
- All other projective symplectic groups are simple. (Proof using transvections and Iwasawa's Lemma as for $PSL_n(q)$.)

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Then another induction on n gives the order of the group as

$$\prod_{i=1}^{n} q^{i-1}(q^i - (-1)^i) = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - (-1)^i).$$

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- The scalars in $SU_n(q)$ form a group of order $(n, q + 1)$.

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- $PSU_3(2) \cong 3^2{:}Q_8$ and $PGU_3(2) \cong 3^2{:}SL_2(3)$
- All other $PSU_n(q)$ are simple.

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- But in odd dimensions, the two types of forms are scalar multiples of each other, so the two groups are the same.
- In even dimensions, $2n$ say, the form has plus type if there is a totally isotropic subspace of dimension n.
- This is not the same as having an orthonormal basis.

Orthogonal groups, odd characteristic

- The orthogonal groups are the isometry groups of non-singular symmetric bilinear forms.
- Since there are two types of forms, there are two types of groups.
- But in odd dimensions, the two types of forms are scalar multiples of each other, so the two groups are the same.
- In even dimensions, $2n$ say, the form has plus type if there is a totally isotropic subspace of dimension n.
- This is not the same as having an orthonormal basis.
- The other forms have minus type, and their maximal totally isotropic subspaces have dimension $n - 1$.

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- The subgroup of index 2 consisting of matrices of determinant 1 is the special orthogonal group.
- The subgroup of scalars has order 2.
- The resulting projective special orthogonal group is NOT simple in general.
- There is (usually) a further subgroup of index 2, which is not so easy to describe.

The spinor norm

- (With some exceptions?) orthogonal groups are generated by reflections:

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- The projective version $P\Omega_n(q)$ is simple, provided $n \geq 5$.

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- The only scalar in the orthogonal group is 1.
- Spinor norms have no meaning.
- But still the orthogonal groups are not simple.

The quasideterminant

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- This subgroup is simple provided $n \geq 6$.

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- $PSO_6^-(q) \cong PSU_4(q).2$, an extension by the field automorphism $x \mapsto x^q$ (applied to each matrix entry, in the case of the standard unitary form).

THE END