Circular designs balanced for neighbours at distances one and two

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Ongoing joint work with Tank Aldred (University of Otago, New Zealand), Brendan McKay (ANU, Australia) and Ian Wanless (Monash University, Australia)

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In one variant of this, self-neighbours are forbidden. In a further variant, it is assumed that the left-neighbour effect is the same as the right-neighbour effect, so all that is needed is that each unordered pair of treatments occurs just once as neighbours and just once with a single unit in between.

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I shall report progress on finding methods of constructing the three types of design.

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Solution: Use a neighbour-balanced design in which each ordered pair (i, j) of different varieties occurs the same number of times as (South, North) neighbours.

An experiment on control of aphids

Entomologists wanted to compare several sprays to deter aphids from the crop without killing them.
The sprays should be applied to a square array of rectangular plots in a single field, using a Latin square (each spray occurs on one plot per row and one plot per column).

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The aphids are as likely to spread East as West, so direction in one dimension is not an issue,
but the North–South effect may be different from the East–West one, because the plots are not square.

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Solution: Use a quasi-complete Latin square,
in which each unordered pair $\{i, j\}$ of sprays occurs
the same number of times as neighbours within rows
and the same number of times as neighbours within columns.

The experiment at Rothamsted on control of aphids

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The marine biologist required that

(i) each ordered pair of items should occur just once as ordered neighbours around the circumference of the tank;

(ii) each ordered pair of items should occur just once with a single item in between them, in order.

A circular design for 5 treatments with neighbour balance at distances one and two

4 1 1
2 3
3 4
2 3
4 0
4 0
0 1
4 0
1 2
2 2
1 0
3 3

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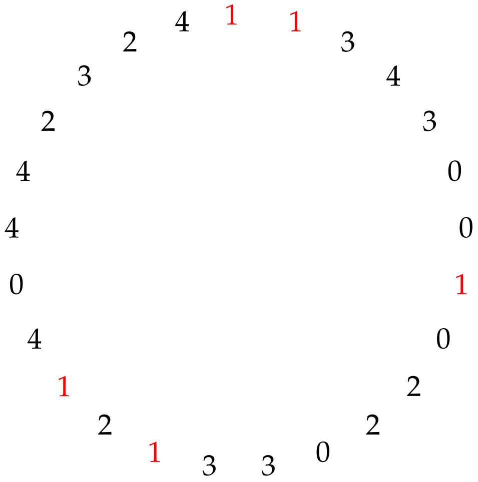

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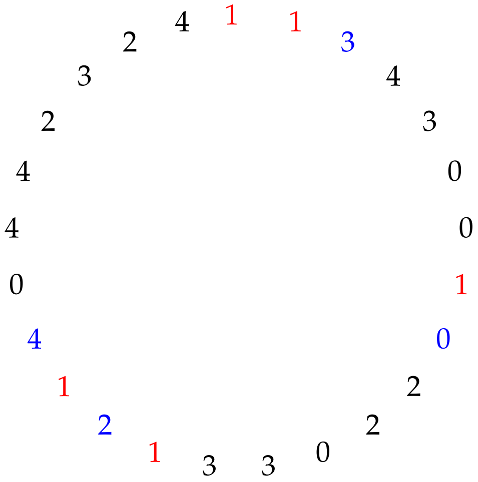

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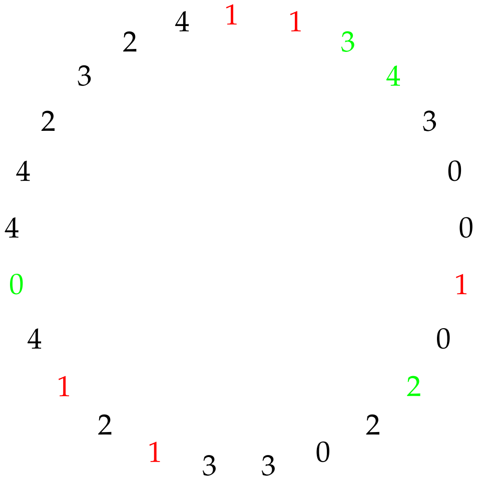

The lazy way to write the design

$$(1\ \ 1\ \ 3\ \ 4\ \ 3\ \ 0\ \ 0\ \ 1\ \ 0\ \ 2\ \ 2\ \ 0\ \ 3\ \ 3\ \ 1\ \ 2\ \ 1\ \ 4\ \ 0\ \ 4\ \ 4\ \ 2\ \ 3\ \ 2\ \ 4)$$

4 1 1
2 3
3 4
2 3
4 0
4 0
0 1
4 0
1 2
2 2
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where the ε_i are independent random variables with mean 0 and common variance σ^2.

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if and only if each pair (λ_j, δ_k) occurs equally often and each pair (δ_j, ρ_k) occurs equally often and each pair (λ_j, ρ_k) occurs equally often;
in other words, the design has neighbour balance at distances one and two.

Those conditions again

Among the triples of the form

$$\big(\tau(i-1), \tau(i), \tau(i+1)\big),$$

each ordered pair of treatments occurs once in positions 1 and 2, once in positions 1 and 3, and once in positions 2 and 3.

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These are conditions for a Latin square whose rows and columns have the same labels as the symbols—a quasigroup.

Building the design from a quasigroup (Latin square)

The quasigroup operation $\circ$ is defined by

$$a \circ b = \text{symbol in row } a \text{ and column } b \text{ of the Latin square.}$$

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We can start with any ordered pair (x, y) and successively build the circular design from the quasigroup as

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

Latin square to circle

$$\begin{array}{c|cccc} \circ & A & B & C & D \\ \hline A & B & A & D & C \\ B & C & D & A & B \\ C & D & C & B & A \\ D & A & B & C & D \end{array}$$

$$(\; A \quad A$$

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This quasigroup gives a design with four separate circles, not one.

$$(\; A \quad A \quad B \quad A \quad C \quad D \;)$$

$$(\; A \quad D \quad C \quad C \quad B \quad C \;)$$

$$(\; B \quad B \quad D \;)$$

$$(\; D \;)$$

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1	2	3	1	4	0
2	3	4	0	2	1
3	0	2	4	1	3
4	4	1	3	0	2

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For every other value of n that we have tried,
we have found an Eulerian quasigroup by computer search;
and we can prove that existence for coprime n and m implies existence for mn;
BUT we have been unable to prove that they always exist.

Exercise

Show that, if $Q = \mathbb{Z}_{p^s}$ or $Q = \mathrm{GF}(p^s)$, then no binary operation of the form

$$x \circ y = ax + by + c$$

makes Q into an Eulerian quasigroup.

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Preece (1976) showed that, for overall balance, the missing pairs at distance two must also be the self-pairs.

Idempotent Eulerian circular sequences

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Our circular design is equivalent to an idempotent quasigroup in which the $n(n-1)$ off-diagonal cells give a single circle.

Construction when $n = 6$ (in general, when n is even)

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Differences at distance one come from the original sequence; most differences at distance two are the neighbour sums.

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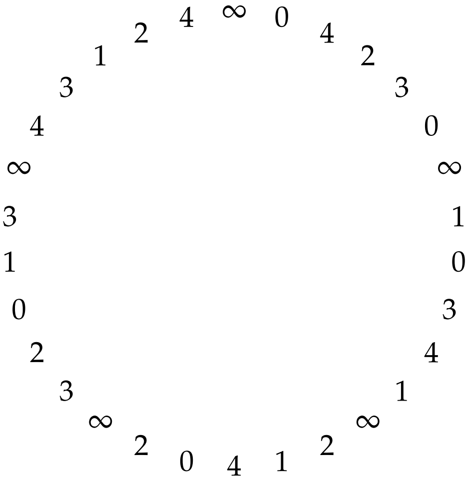

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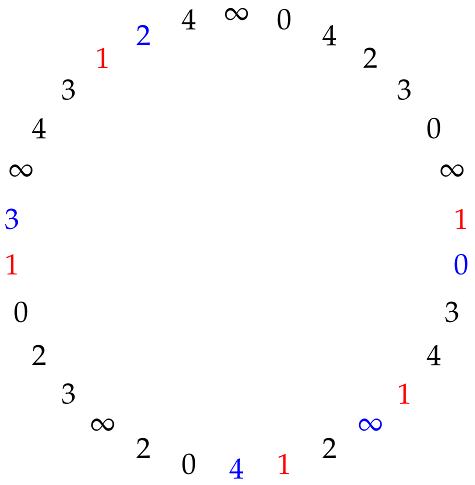

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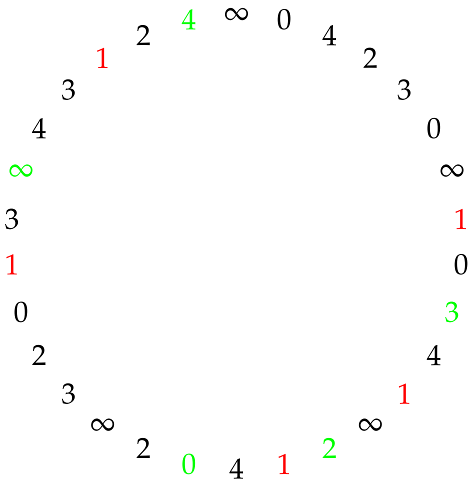

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Given an initial sequence of the non-zero integers modulo $n-1$ satisfying those conditions, that construction always produces an idempotent Eulerian circular sequence.

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Any triple (a, b, a) gives b as a neighbour of a on both sides, so there can be no such triples.

Model and variance

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In vector form

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Some calculations show that the variance of the estimator of the difference between two direct effects is

$$\frac{2(2r-1)}{(r-1)(2r+1)}\sigma^2$$

while that for the difference between two neighbour effects is

$$\frac{2r}{(r-1)(2r+1)}\sigma^2.$$

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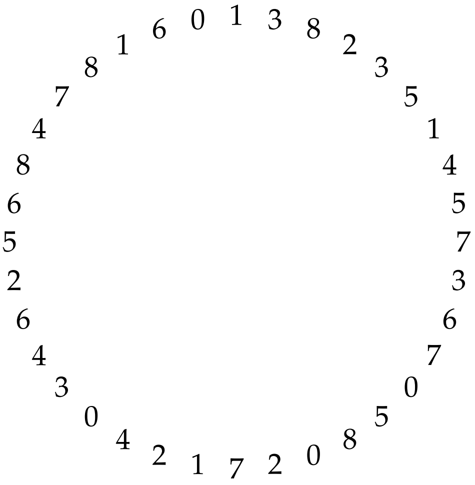
A circular design for 9 treatments with undirectional neighbour balance at distances one and two



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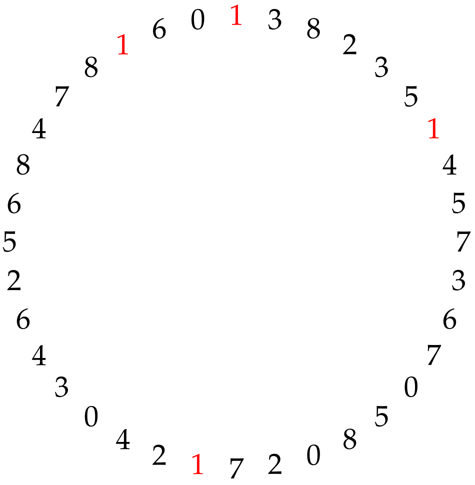

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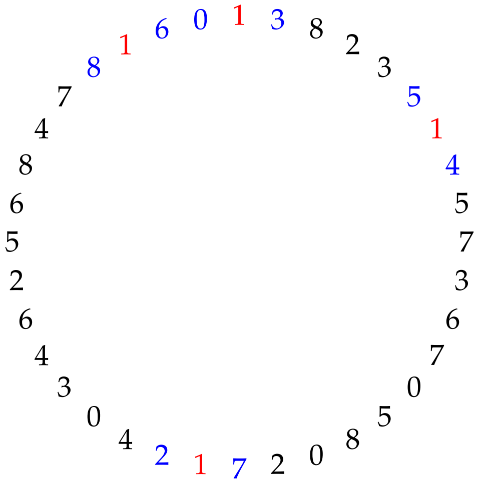

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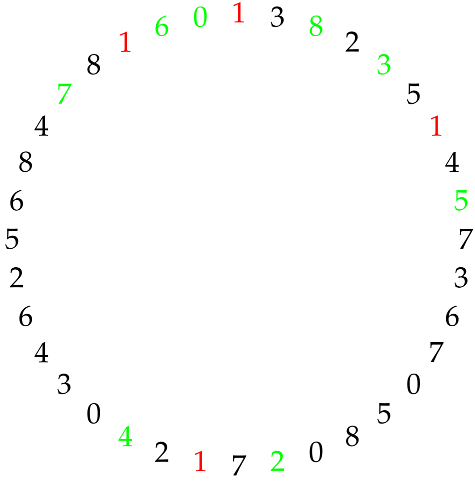

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Solution for variant II

Theorem

Given an initial circular sequence of $(n-1)/2$ of the integers modulo n satisfying those conditions, that construction always produces a circular sequence balanced for undirected neighbours at distances one and two.

Theorem

Such an initial sequence can be constructed whenever n is odd and $n \geq 9$. There is also such a circular sequence when $n = 7$.

Back to the original question

A quasigroup of order n with operation $\circ$ is Eulerian if the sequence

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

does not repeat before n^2 steps.

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Conjecture

If $n \geq 5$ then there exists an Eulerian quasigroup of order n.

Coprime sizes

Theorem

If Q_1 and Q_2 are Eulerian quasigroups of orders n and m, where n and m are coprime, then $Q_1 \otimes Q_2$ is an Eulerian quasigroup of order nm.

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Proof.

In the sequence

$$(a, x) \quad (b, y) \quad (a \Box b, x \circ y) \quad (b \Box (a \Box b), y \circ (x \circ y)) \quad \cdots$$

the first coordinates repeat every n^2 steps, but not earlier, and the second coordinates repeat every m^2 steps, but not earlier. $\square$

Strategy

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- powers of 2 bigger than 4.

Homework

If $Q = \mathbb{Z}_{p^s}$ or $Q = \mathrm{GF}(p^s)$, then no binary operation of the form

$$x \circ y = ax + by + c$$

makes Q into an Eulerian quasigroup.

Technique

If q is odd, try taking $Q = \mathbb{Z}_q$ and putting

$$x \circ y = \pi(x + y)$$

where π is a relatively simple permutation.

Technique

If q is odd, try taking $Q = \mathbb{Z}_q$ and putting

$$x \circ y = \pi(x + y)$$

where π is a relatively simple permutation.

For example, when $q = 7$ put $\pi = (0\,1\,2)(3\,4)$ so that

$$4 \circ 5 = \pi(4 + 5) = \pi(2) = 0.$$

Obstacle

Theorem

If n is even then no Eulerian quasigroup can be obtained from a group of order n by permutions of rows, columns or symbols.

...for all practical purposes

Theorem

If $n \geq 5$ and there is no Eulerian quasigroup of order n then n is divisible by a prime power exceeding 500.