Structure balance for experiments which are randomized in stages

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joint work with C. J. Brien, University of South Australia

26 April 2007

Overall framework

- Randomization with three or more tiers (JRSSB, 2006)

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- Analysis

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- (Randomization) model
- Analysis
- Advice on design

Overall framework

- Randomization with three or more tiers (JRSSB, 2006)
- (Randomization) model
- Analysis
- Advice on design (JRSSA, 1981)

Overall framework

- Randomization with three or more tiers (JRSSB, 2006)
- Orthogonal decomposition of the data space (submitted)
- (Randomization) model (in progress)
- Analysis
- Advice on design (JRSSA, 1981)

Three experiments described by R. F. White (1975)

A clinical trial

2 treatments | 2 Therapies → 10 Doctors, 6 Patients in D | 60 patients

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Laboratory animals

2 Drugs → 10 Days → 60 Animals

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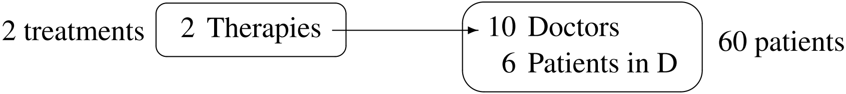
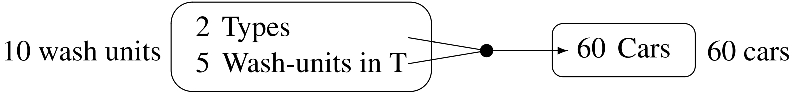


Laboratory animals

A car-wash emporium



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Doctors	9
Patients[Doctors]	50

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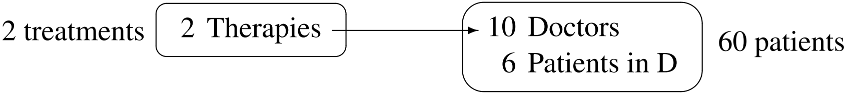

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Doctors	9	Therapies	1
		Residual	8
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A clinical trial: skeleton anova

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The skeleton analysis of variance shows

- which "block" term the Therapies term is confounded with, hence the likely magnitude of the variance of the estimator of the contrast between the two therapies

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- the relevant residual term, and its degrees of freedom, hence the likely precision of the estimator of the variance.

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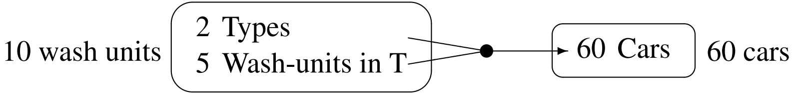
"Anova—why it is as easy as ever", RAB & SGG

A car-wash emporium: skeleton anova



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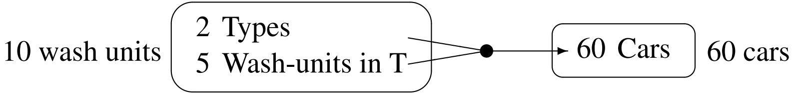

cars tier		
source	df	
Mean	1	
Cars	59	

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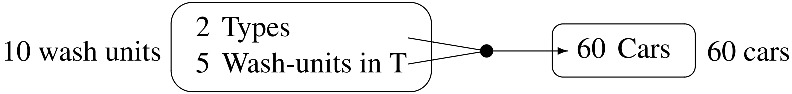
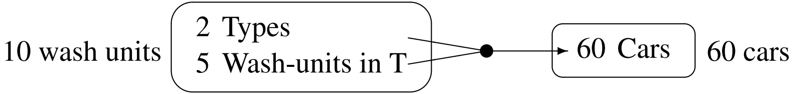


cars tier		washunits tier	
source	df	source	df
Mean	1		
Cars	59		

A car-wash emporium: skeleton anova



cars tier		washunits tier	
source	df	source	df
Mean	1	Mean	1
Cars	59		

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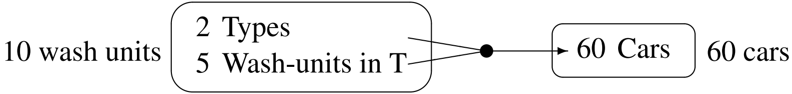

cars tier		washunits tier	
source	df	source	df
Mean	1	Mean	1
Cars	59	Types	1

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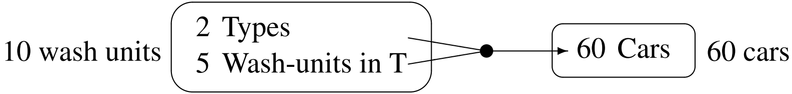

cars tier		washunits tier	
source	df	source	df
Mean	1	Mean	1
Cars	59	Types	1
		Washunits[Types]	8

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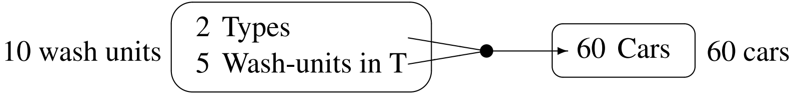

cars tier		washunits tier	
source	df	source	df
Mean	1	Mean	1
Cars	59	Types	1
		Washunits[Types]	8
		Residual	50

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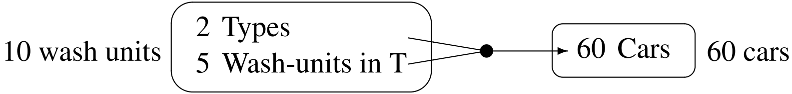

cars tier		washunits tier	
source	df	source	df
Mean	1	Mean	1
Cars	59	Types	1
		Washunits[Types]	8
		Residual	50

Now we can discuss whether Washunits[Types] is fixed or random.

Laboratory animals: skeleton anova

2 Drugs → 10 Days → 60 Animals

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animals tier			
source	df		
Mean	1		
Animals	59		

Laboratory animals: skeleton anova

2 Drugs → 10 Days → 60 Animals

animals tier		days tier		
source	df	source	df	
Mean	1			
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Laboratory animals: skeleton anova

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animals tier		days tier		
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animals tier		days tier		treatments tier	
source	df	source	df	source	df
Mean	1	Mean	1		
Animals	59	Days	9		
		Residual	50		

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2 Drugs $\rightarrow$ 10 Days $\rightarrow$ 60 Animals

animals tier		days tier		treatments tier	
source	df	source	df	source	df
Mean	1	Mean	1	Mean	1
Animals	59	Days	9		
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animals tier		days tier		treatments tier	
source	df	source	df	source	df
Mean	1	Mean	1	Mean	1
Animals	59	Days	9	Drugs	1
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animals tier		days tier		treatments tier	
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- Differences between Drugs are confounded with differences between Days and differences between Animals.

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Mean	1	Mean	1	Mean	1
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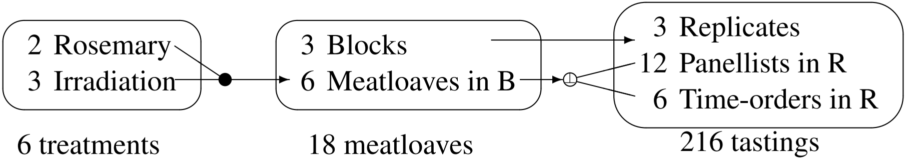
- Differences between Drugs are confounded with differences between Days and differences between Animals.
- The 8-df Residual gives the variability between Days plus the variability between Animals.

Meatloaves: T. B. Bailey



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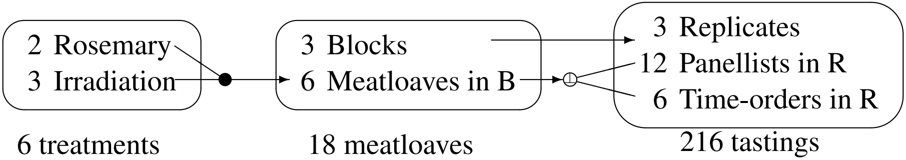

tastings tier		meatloaves tier		treatments tier	
source	df	source	df	source	df
Mean	1	Mean	1	Mean	1
Replicates	2	Blocks	2		
Panellists[Reps]	33				
Time-orders[Reps]	15				
P#T[Reps]	165	Meatloaves[B]	15	Rosemary	1
				Irradiation	2
				R# I	2
				Residual	10
		Residual	150		

Basic idea

set Γ of objects

set Ω of objects

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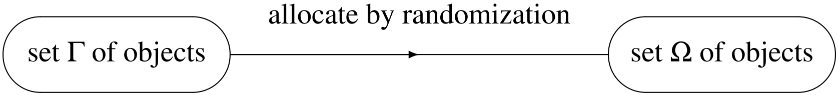

Basic idea

$$\text{set } \Gamma \text{ of objects} \xrightarrow{\text{allocate by randomization}} \text{set } \Omega \text{ of objects}$$

vector space $V_\Gamma = \mathbb{R}^\Gamma$

vector space $V_\Omega = \mathbb{R}^\Omega$

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Decomposition $\mathscr{Q}$ of V_Γ into orthogonal subspaces

Decomposition $\mathscr{P}$ of V_Ω into orthogonal subspaces

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$\mathscr{Q}$ further decomposes $\mathscr{P}$

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- What happens if the design (allocation) is not orthogonal?

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Decomposition $\mathscr{Q}$ of V_Γ into orthogonal subspaces

Decomposition $\mathscr{P}$ of V_Ω into orthogonal subspaces

$\mathscr{Q}$ further decomposes $\mathscr{P}$

- What happens if the design (allocation) is not orthogonal?
- What happens if there are 2 or more stages of (random) allocation?

A pair of orthogonal decompositions

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$\equiv$ a set of real $\Omega \times \Omega$ matrices $\mathbf{P}_1, \ldots, \mathbf{P}_n$ which

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Given an allocation of Γ to Ω, we can regard subspaces of V_Γ as subspaces of V_Ω and hence regard $\mathbf{Q}_1, \ldots, \mathbf{Q}_m$ as $\Omega \times \Omega$ matrices.

A pair of orthogonal decompositions: continued

$$
\begin{aligned}
\mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n &= \mathbf{I} \quad \text{(identity for } V_\Omega\text{)} \\
\mathbf{Q}_1 + \mathbf{Q}_2 + \cdots + \mathbf{Q}_m &= \mathbf{I}_{\mathscr{Q}} \quad \text{(identity for } V_\Gamma \text{ in } V_\Omega\text{)}
\end{aligned}
$$

A pair of orthogonal decompositions: continued

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The design is orthogonal if
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that is, for each $\mathbf{Q}_i$ there is some j such that

- $\mathbf{Q}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{Q}_i = \mathbf{Q}_i$
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A pair of orthogonal decompositions: continued

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All designs in the preceding examples are orthogonal.

Structure balance (following Nelder)

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\end{array}
$$

Definition

A structure $\mathscr{Q}$ is structure-balanced in relation to a structure $\mathscr{P}$ if there are scalars $\lambda_{\mathbf{PQ}}$ for $\mathbf{P}$ in $\mathscr{P}$ and $\mathbf{Q}$ in $\mathscr{Q}$ such that

(i) $\mathbf{QPQ} = \lambda_{\mathbf{PQ}}\mathbf{Q}$ for all $\mathbf{P}$ in $\mathscr{P}$ and all $\mathbf{Q}$ in $\mathscr{Q}$, and

(ii) $\mathbf{Q}_1\mathbf{P}\mathbf{Q}_2 = \mathbf{0}$ for all $\mathbf{P}$ in $\mathscr{P}$ and all $\mathbf{Q}_1 \neq \mathbf{Q}_2$ in $\mathscr{Q}$.

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The structure $\mathscr{Q}$ is orthogonal in relation to $\mathscr{P}$ if (i) and (ii) hold with each $\lambda_{\mathbf{PQ}}$ equal to either 1 or 0.

Structure balance (following Nelder)

$$\begin{array}{cccll} \mathscr{P} & \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n & = & \mathbf{I} & \text{(identity for } V_\Omega\text{)} \\ \mathscr{Q} & \mathbf{Q}_1 + \mathbf{Q}_2 + \cdots + \mathbf{Q}_m & = & \mathbf{I}_{\mathscr{Q}} & \text{(identity for } V_\Gamma \text{ in } V_\Omega\text{)} \end{array}$$

Definition

A structure $\mathscr{Q}$ is structure-balanced in relation to a structure $\mathscr{P}$ if there are scalars $\lambda_{\mathbf{PQ}}$ for $\mathbf{P}$ in $\mathscr{P}$ and $\mathbf{Q}$ in $\mathscr{Q}$ such that

(i) $\mathbf{QPQ} = \lambda_{\mathbf{PQ}}\mathbf{Q}$ for all $\mathbf{P}$ in $\mathscr{P}$ and all $\mathbf{Q}$ in $\mathscr{Q}$, and

(ii) $\mathbf{Q}_1\mathbf{P}\mathbf{Q}_2 = \mathbf{0}$ for all $\mathbf{P}$ in $\mathscr{P}$ and all $\mathbf{Q}_1 \neq \mathbf{Q}_2$ in $\mathscr{Q}$.

The structure $\mathscr{Q}$ is orthogonal in relation to $\mathscr{P}$ if (i) and (ii) hold with each $\lambda_{\mathbf{PQ}}$ equal to either 1 or 0.

The $\lambda_{\mathbf{PQ}}$ are called efficiency factors and are summarized in the $\mathscr{P} \times \mathscr{Q}$ efficiency matrix $\Lambda_{\mathscr{P}\mathscr{Q}}$.

Structure balance (using James and Wilkinson)

Fix $\mathbf{P}$ and $\mathbf{Q}$.

- $\mathbf{QPQ} = \lambda_{\mathbf{PQ}}\mathbf{Q}$ means that every vector in $\mathrm{Im}(\mathbf{Q})$ makes angle $\cos^{-1}(\lambda_{\mathbf{PQ}})$ with $\mathrm{Im}(\mathbf{P})$.

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Fix $\mathbf{Q}$, let $\mathbf{P}$ vary.

- $\sum \mathbf{P} = \mathbf{I}$ implies that $\sum_{\mathbf{P}} \lambda_{\mathbf{PQ}} = 1$.

Structure balance

Fix $\mathbf{P}$, let $\mathbf{Q}$ vary.

- $\mathbf{Q}_1\mathbf{P}\mathbf{Q}_2 = \mathbf{0}$ implies that $\mathbf{P}(\mathrm{Im}(\mathbf{Q}_1)) \perp \mathbf{P}(\mathrm{Im}(\mathbf{Q}_2))$, so $\mathbf{P} \triangleright \mathbf{Q}_1$ and $\mathbf{P} \triangleright \mathbf{Q}_2$ correspond to orthogonal subspaces of $\mathrm{Im}(\mathbf{P})$.

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- Write $\mathbf{P} \vdash \mathscr{Q} = \mathbf{P} - \sum_{\mathbf{Q}} \mathbf{P} \rhd \mathbf{Q}$, so that $\mathbf{P} \vdash \mathscr{Q}$ corresponds to $\mathrm{Im}(\mathbf{P}) \cap V_\Gamma^\perp$.

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So $\mathbf{P} \triangleright \mathbf{Q}_1, \mathbf{P} \triangleright \mathbf{Q}_2, \ldots, \mathbf{P} \triangleright \mathbf{Q}_m, \mathbf{P} \vdash \mathscr{Q}$ decompose $\mathbf{P}$ orthogonally.

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$$\mathscr{P} \rhd \mathscr{Q} = \{\mathbf{P} \rhd \mathbf{Q} : \mathbf{P} \in \mathscr{P},\ \mathbf{Q} \in \mathscr{Q},\ \lambda_{\mathbf{PQ}} \neq 0\} \cup \{\mathbf{P} \vdash \mathscr{Q} : \mathbf{P} \in \mathscr{P}\}.$$

Toy example from micorarrays

	slides							
	1	2	3	4	5	6	7	8
red	0	1	0	2	0	3	0	4
green	1	0	2	0	3	0	4	0

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	slides							
	1	2	3	4	5	6	7	8
red	0	1	0	2	0	3	0	4
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observations		
source	df	
Mean	1	
Colours	1	
Slides	7	
Colours # Slides	7	

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observations		treatments		
source	df	eff	source	df
Mean	1	1	Mean	1
Colours	1			
Slides	7	$\frac{1}{2}$	Within-rest	3
			Residual	4
Colours # Slides	7	1	Control-v-rest	1
		$\frac{1}{2}$	Within-rest	3
			Residual	3

Structure balance when one arrow follows another

$$\mathscr{R} \longrightarrow \mathscr{Q} \longrightarrow \mathscr{P}$$

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Theorem

If $\mathscr{Q}$ is structure-balanced in relation to $\mathscr{P}$
(with efficiency matrix $\Lambda_{\mathscr{P}\mathscr{Q}}$)
and $\mathscr{R}$ is structure-balanced in relation to $\mathscr{Q}$
(with efficiency matrix $\Lambda_{\mathscr{Q}\mathscr{R}}$)
then

- $\mathscr{R}$ is structure-balanced in relation to $\mathscr{P}$ and $\Lambda_{\mathscr{P}\mathscr{R}} = \Lambda_{\mathscr{P}\mathscr{Q}}\Lambda_{\mathscr{Q}\mathscr{R}}$;

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- $\mathscr{Q} \triangleright \mathscr{R}$ is structure-balanced in relation to $\mathscr{P}$ (and $\Lambda \ldots$);
- $(\mathscr{P} \triangleright \mathscr{Q}) \triangleright \mathscr{R} = \mathscr{P} \triangleright (\mathscr{Q} \triangleright \mathscr{R})$.

Structure balance when one arrow follows another: two uses

$$\mathscr{R} \longrightarrow \mathscr{Q} \longrightarrow \mathscr{P}$$

Composed randomizations

These can be done in either order. No knowledge of of the outcome of one is needed to perform the other.

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Randomized-inclusive randomizations

The outcome of the randomization $\mathscr{R} \longrightarrow \mathscr{Q}$ must be known before the randomization $\mathscr{Q} \longrightarrow \mathscr{P}$ can be performed,
because the design for $\mathscr{Q} \longrightarrow \mathscr{P}$ needs information about $\mathscr{R}$.
Typically, this information in encoded in pseudofactors for $\mathscr{Q}$.
Typically, the original $\mathscr{Q} \triangleright \mathscr{R}$ is not informative enough.

Structure balance when one arrow follows another: the randomization model

$$\mathscr{R} \longrightarrow \mathscr{Q} \longrightarrow \mathscr{P}$$

Good news

If the allowable permutations of the set of objects admitting $\mathscr{P}$ give a covariance matrix with eigenprojectors $\mathbf{P}_1, \ldots, \mathbf{P}_n$

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then the combined randomization gives a covariance matrix whose eigenprojectors are $\mathscr{P} \rhd \mathscr{Q}$.

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The 'stratum variances' may satsify some linear inequalities
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ANOVA assumes they are unrelated.
REML has problems.

Structure balance when two arrows have the same end (1)

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There are three possibilities.

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Unrandomized-inclusive randomizations

The outcome of the randomization $\mathscr{Q} \longrightarrow \mathscr{P}$ is known;
the design for $\mathscr{R} \longrightarrow \mathscr{P}$ and method of randomizing $\mathscr{R} \longrightarrow \mathscr{P}$ both use knowledge of $\mathscr{P} \rhd \mathscr{Q}$.

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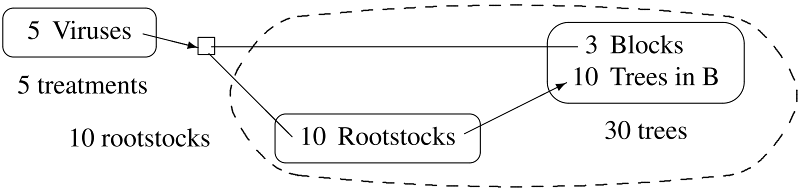
the design for $\mathscr{R} \longrightarrow \mathscr{P}$ and method of randomizing $\mathscr{R} \longrightarrow \mathscr{P}$ both use knowledge of $\mathscr{P} \rhd \mathscr{Q}$.
Assume that $\mathscr{Q}$ is structure-balanced in relation to $\mathscr{P}$,
and that $\mathscr{R}$ is structure-balanced in relation to $\mathscr{P} \rhd \mathscr{Q}$.
Use the decompostion $(\mathscr{P} \rhd \mathscr{Q}) \rhd \mathscr{R}$.

Superimposed Experiment in a Row-Column Design



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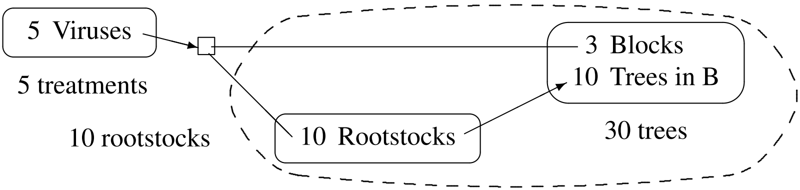

trees tier		rootstocks tier		treatments tier		
source	df	source	df	eff	source	df
Mean	1	Mean	1		Mean	1
Blocks	2					
Trees[Blocks]	27	Rootstocks	9	$\frac{1}{6}$	Viruses	4
					Residual	5
		Residual	18	$\frac{5}{6}$	Viruses	4
					Residual	14

Randomization model for unrandomized-inclusive randomizations

There are a lot of possibilities.

Randomization model for unrandomized-inclusive randomizations

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How much notice should we take of a randomization performed 20 years ago?

Randomization model for unrandomized-inclusive randomizations

There are a lot of possibilities.

How much notice should we take of a randomization performed 20 years ago?

Not all possibilities give stratifiable covariance matrices.

Structure balance when two arrows have the same end (2)

$$\mathscr{Q} \longrightarrow \mathscr{P} \longleftarrow \mathscr{R}$$

Independent randomizations

The two structurally balanced designs are chosen so that,
for all **P** except the Mean,
either every **PQ** is zero or every **PR** is zero.
Thus $\mathscr{Q}$ and $\mathscr{R}$ do not interfere with each other in $\mathscr{P}$.

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- $\mathscr{Q}$ is structure-balanced in relation to $\mathscr{P} \rhd \mathscr{R}$;
- $\mathscr{R}$ is structure-balanced in relation to $\mathscr{P} \rhd \mathscr{Q}$;

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- $\mathscr{Q}$ is structure-balanced in relation to $\mathscr{P} \rhd \mathscr{R}$;
- $\mathscr{R}$ is structure-balanced in relation to $\mathscr{P} \rhd \mathscr{Q}$;
- $(\mathscr{P} \rhd \mathscr{R}) \rhd \mathscr{Q} = (\mathscr{P} \rhd \mathscr{Q}) \rhd \mathscr{R}$.

Structure balance when two arrows have the same end (3)

$$\mathscr{Q} \longrightarrow \mathscr{P} \longleftarrow \mathscr{R}$$

Coincident randomizations

The two structurally balanced designs are chosen so that, for all $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$

- either $\mathbf{PQ}$ is zero (after $\mathbf{P}$, ignore $\mathbf{Q}$)
- or $\mathbf{PR}$ is zero (after $\mathbf{P}$, ignore $\mathbf{R}$)
- or $\mathbf{P} \rhd \mathbf{Q} = \mathbf{P}$ (after $\mathbf{P}$, do $\mathbf{Q}$ before $\mathbf{R}$)
- or $\mathbf{P} \rhd \mathbf{R} = \mathbf{P}$ (after $\mathbf{P}$, do $\mathbf{R}$ before $\mathbf{Q}$)

Structure balance when two arrows have the same end (4)

$$\mathscr{Q} \longrightarrow \mathscr{P} \longleftarrow \mathscr{R}$$

Theorem

Given a pair of structurally balanced randomizations which are independent or coincident (or also unrandomized-inclusive if $V_\Gamma \cap \mathrm{Mean}^\perp$ is orthogonal to $V_\Upsilon \cap \mathrm{Mean}^\perp$), the decompositions $\mathscr{P} \rhd \mathscr{Q}$ and $\mathscr{P} \rhd \mathscr{R}$ are compatible in the sense that if $\mathbf{A} \in \mathscr{P} \rhd \mathscr{Q}$ and $\mathbf{B} \in \mathscr{P} \rhd \mathscr{R}$ then $\mathbf{AB} = \mathbf{BA}$. Hence

$$\{\mathbf{AB} : \mathbf{A} \in \mathscr{P} \rhd \mathscr{Q},\ \mathbf{B} \in \mathscr{P} \rhd \mathscr{R}\}$$

gives an orthogonal decomposition of V_Ω.

Randomization model for independent or coincident randomizations

Everything works . . .

Randomization model
for independent or coincident randomizations

Everything works …

… but there are no inter-tier interactions.

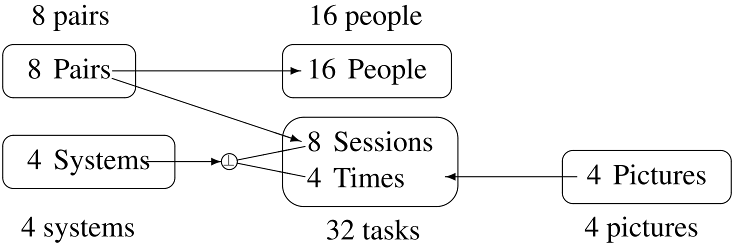
Randomization model for independent or coincident randomizations

Everything works …

… but there are no inter-tier interactions.

If you want $\mathscr{Q}$-$\mathscr{R}$ interactions, you should use a single randomization.

Telephone systems: Lewis and Russell (1998)



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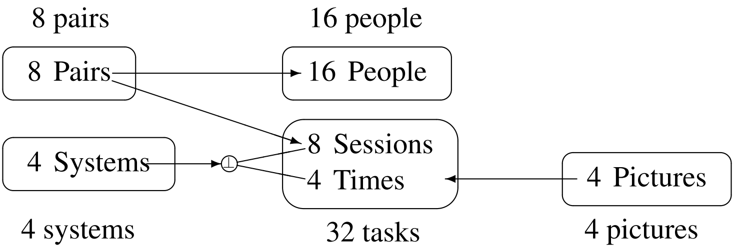

tasks tier		pairs tier		pictures tier		systems tier	
Mean	1	Mean	1	Mean	1	Mean	1
Sessions	7	Pairs	7				
Times	3			Pictures	3		
Sessions # Times	21					systems	3
						residual	18