Using characters of Abelian groups, (and the design key), to construct designs for experiments in glasshouses

R. A. Bailey

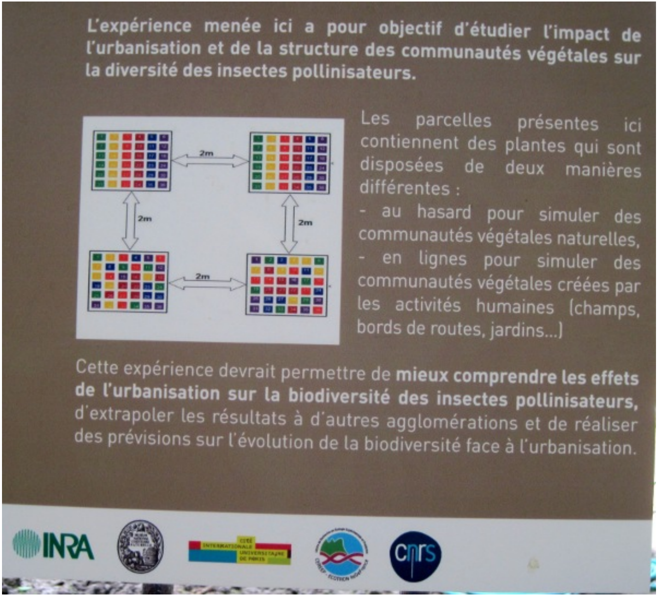
Queen Mary University of London

r.a.bailey@qmul.ac.uk

Joint work with Chris Brien and Thao Tran, University of South Australia

This talk is dedicated to André Kobilinsky

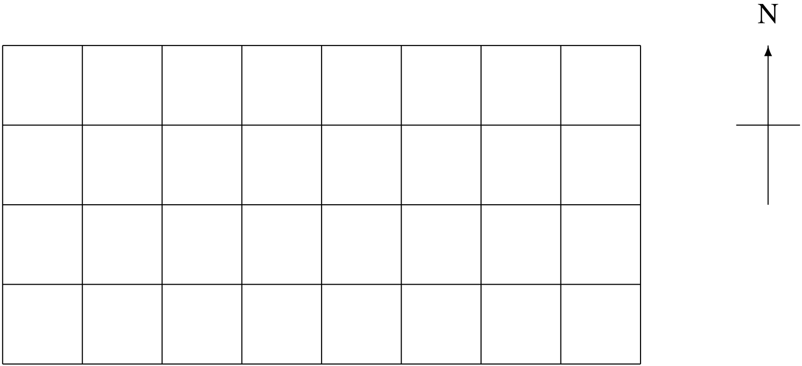
Jardin des Plantes, Mai 2011



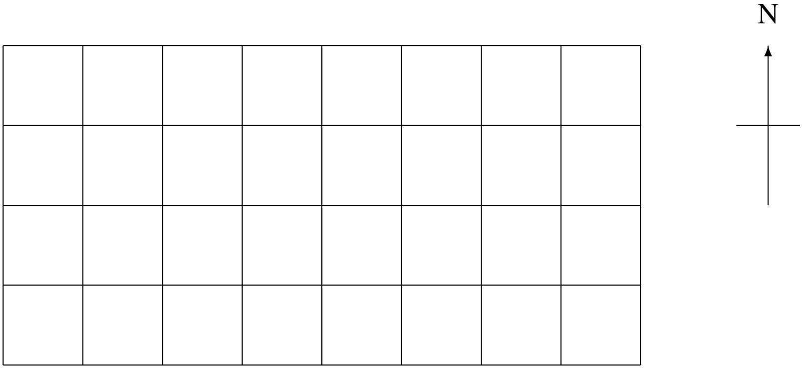
Waite Institute, Adelaide, Australia, 2010

Indian Agricultural Research Institute, Delhi, 2006

A glasshouse



A glasshouse



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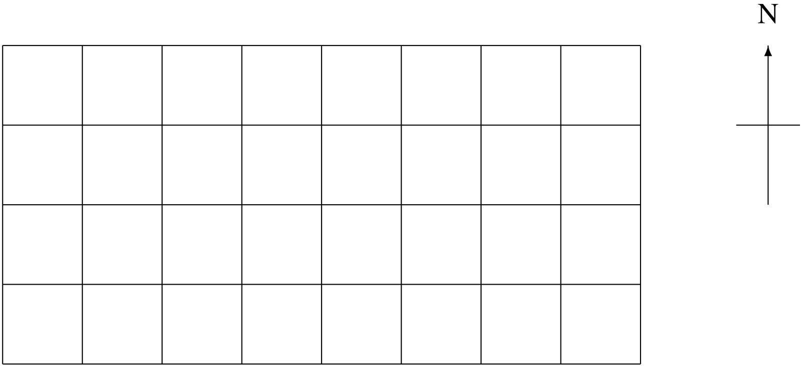
A glasshouse



Glasshouses often have their axes aligned North–South and East–West (for example, Les Serres D'Auteuil),
so experiments in glasshouses should use rows and columns as blocks.

Aux Serres D'Auteuil, Avril 2011

Quasi-Latin square designs

A quasi-Latin square is a square containing one or more complete replicates of the treatments, where no row or column contains all treatments.

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1	2	3	4
5	6	7	8
3	8	1	6
7	4	5	2

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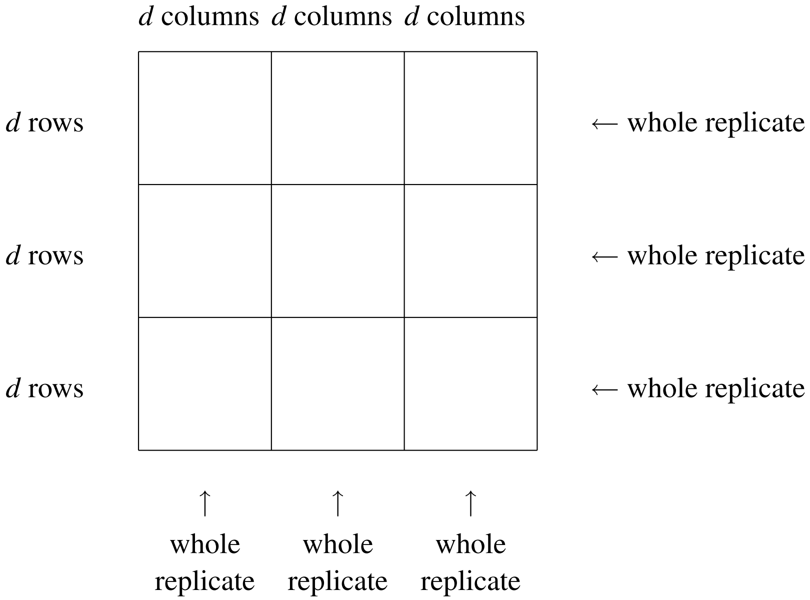
A quasi-Latin square design consists of one or more quasi-Latin squares.

Suppose that we have s squares of size $k \times k$;
that we have v treatments;
that $vr = k^2$;
and that $v = dk$, so that $k = dr$.

Idea of Construction: k rows and k columns, dk treatments



Characters, working modulo 3, for 3×3 factorial

Characters	Treatments								
Factor A	0	0	0	1	1	1	2	2	2
Factor B	0	1	2	0	1	2	0	1	2
$A+B$	0	1	2	1	2	0	2	0	1
$A+2B$	0	2	1	1	0	2	2	1	0
$2A+B$	0	1	2	2	0	1	1	2	0
$2A+2B$	0	2	1	2	1	0	1	0	2
$2A$	0	0	0	2	2	2	1	1	1
$2B$	0	2	1	0	2	1	0	2	1
I	0	0	0	0	0	0	0	0	0

Characters, working modulo 3, for 3×3 factorial

Characters		Treatments								
Factor	A	0	0	0	1	1	1	2	2	2
Factor	B	0	1	2	0	1	2	0	1	2
$A+B$		0	1	2	1	2	0	2	0	1
$A+2B$		0	2	1	1	0	2	2	1	0
$2A+B$		0	1	2	2	0	1	1	2	0
$2A+2B$		0	2	1	2	1	0	1	0	2
$2A$		0	0	0	2	2	2	1	1	1
$2B$		0	2	1	0	2	1	0	2	1
I		0	0	0	0	0	0	0	0	0

$A \equiv 2A$	main effect of A
$B \equiv 2B$	main effect of B
$A+B \equiv 2A+2B$	2 degrees of freedom for the A-by-B interaction
$A+2B \equiv 2A+B$	2 degrees of freedom for the A-by-B interaction, orthogonal to the previous 2

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$A+B$	0	1	2	1	2	0	2	0	1
$A+2B$	0	2	1	1	0	2	2	1	0
$2A+B$	0	1	2	2	0	1	1	2	0
$2A+2B$	0	2	1	2	1	0	1	0	2
$2A$	0	0	0	2	2	2	1	1	1
$2B$	0	2	1	0	2	1	0	2	1
I	0	0	0	0	0	0	0	0	0

$A \equiv 2A$	main effect of A
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For 3 blocks of size 3, can alias blocks with any character.

Characters, working modulo 2, for $2 \times 2 \times 2$ factorial

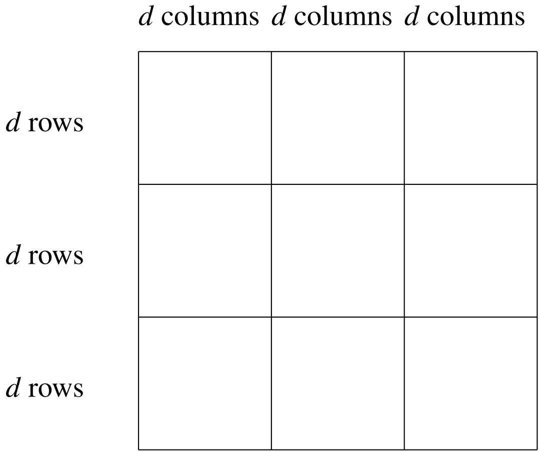
Characters	Treatments							
Factor A	0	0	0	0	1	1	1	1
Factor B	0	0	1	1	0	0	1	1
Factor C	0	1	0	1	0	1	0	1
$A+B$	0	0	1	1	1	1	0	0
$A+C$	0	1	0	1	1	0	1	0
$B+C$	0	1	1	0	0	1	1	0
$A+B+C$	0	1	1	0	0	1	1	0
I	0	0	0	0	0	0	0	0

Characters, working modulo 2, for $2 \times 2 \times 2$ factorial

Characters		Treatments							
Factor	A	0	0	0	0	1	1	1	1
Factor	B	0	0	1	1	0	0	1	1
Factor	C	0	1	0	1	0	1	0	1
	$A+B$	0	0	1	1	1	1	0	0
	$A+C$	0	1	0	1	1	0	1	0
	$B+C$	0	1	1	0	0	1	1	0
	$A+B+C$	0	1	1	0	0	1	1	0
	I	0	0	0	0	0	0	0	0

A	main effect of A
B	main effect of B
C	main effect of C
$A+B$	(1 degree of freedom for) the A-by-B interaction
$A+C$	(1 degree of freedom for) the A-by-C interaction
$B+C$	(1 degree of freedom for) the B-by-C interaction
$A+B+C$	(1 degree of freedom for) the A-by-B-by-C interaction

p^t rows and columns, p^m treatments, $d = p^{m-t}$



p^t rows and columns, p^m treatments, $d = p^{m-t}$

	d columns	d columns	d columns	
d rows				$\leftarrow$ confound $d-1$ df with rows using 'row characters'
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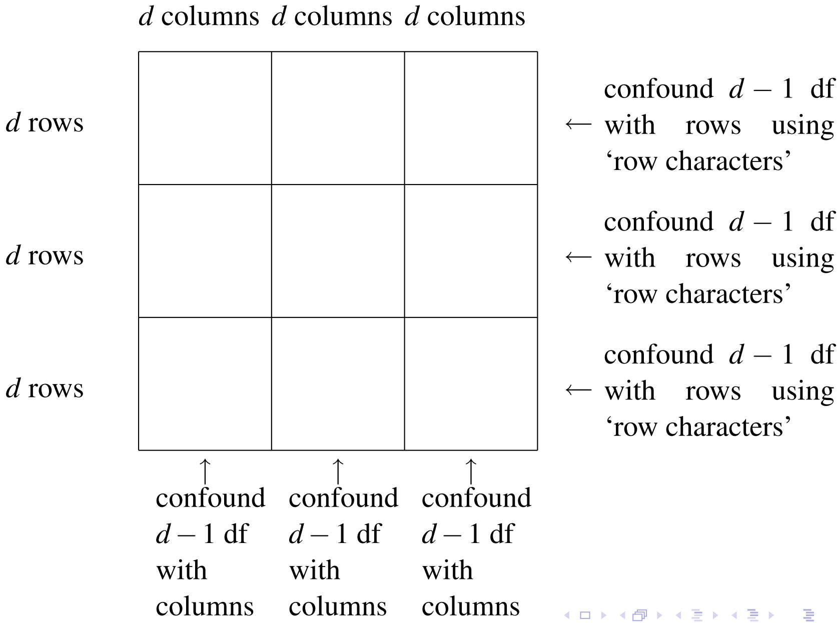
p^t rows and columns, p^m treatments, $d = p^{m-t}$



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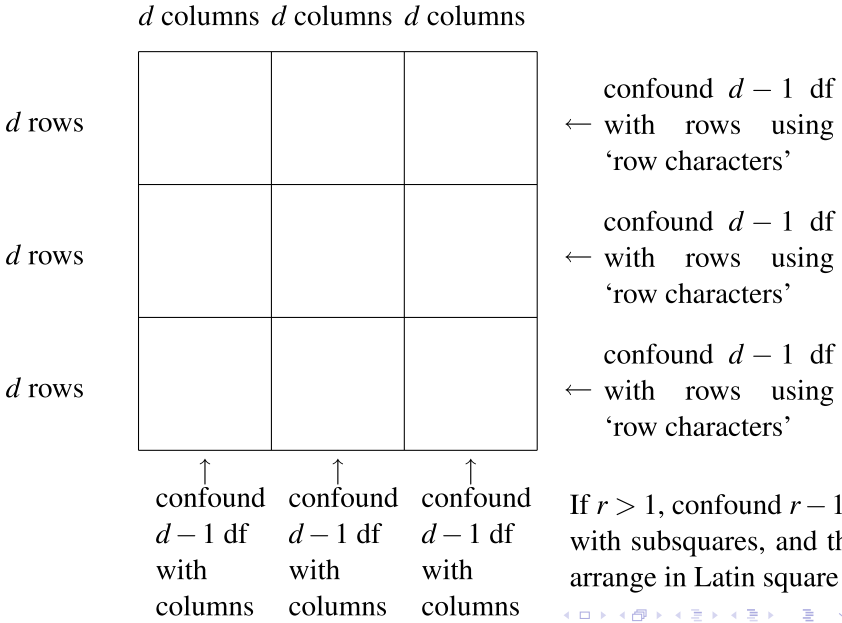

If $r > 1$, confound $r-1$ df with subsquares, and then arrange in Latin square

A 2^3 factorial experiment in 4 rows and 4 columns

$B+C=0$				
$B+C=1$				

A 2^3 factorial experiment in 4 rows and 4 columns

$B+C=0$				
$B+C=1$				
$A+B+C=0$				
$A+B+C=1$				

A 2^3 factorial experiment in 4 rows and 4 columns

	$A+B=0$	$A+B=1$		
$B+C=0$				
$B+C=1$				
$A+B+C=0$				
$A+B+C=1$				

A 2^3 factorial experiment in 4 rows and 4 columns

	$A+B=0$	$A+B=1$	$A+C=0$	$A+C=1$
$B+C=0$				
$B+C=1$				
$A+B+C=0$				
$A+B+C=1$				

A 2^3 factorial experiment in 4 rows and 4 columns

	$A+B=0$	$A+B=1$	$A+C=0$	$A+C=1$
$B+C=0$				
$B+C=1$				
$A+B+C=0$				
$A+B+C=1$				

$A=1$	$A=0$
$A=0$	$A=1$

A 2^3 factorial experiment in 4 rows and 4 columns

	$A+B=0$	$A+B=1$	$A+C=0$	$A+C=1$
$B+C=0$	$(1,1,1)$	$(1,0,0)$	$(0,0,0)$	$(0,1,1)$
$B+C=1$	$(1,1,0)$	$(1,0,1)$	$(0,1,0)$	$(0,0,1)$
$A+B+C=0$	$(0,0,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$
$A+B+C=1$	$(0,0,1)$	$(0,1,0)$	$(1,1,1)$	$(1,0,0)$

$A=1$	$A=0$
$A=0$	$A=1$

A 2^3 factorial experiment in 4 rows and 4 columns

	$A+B=0$	$A+B=1$	$A+C=0$	$A+C=1$
$B+C=0$	$(1,1,1)$	$(1,0,0)$	$(0,0,0)$	$(0,1,1)$
$B+C=1$	$(1,1,0)$	$(1,0,1)$	$(0,1,0)$	$(0,0,1)$
$A+B+C=0$	$(0,0,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$
$A+B+C=1$	$(0,0,1)$	$(0,1,0)$	$(1,1,1)$	$(1,0,0)$

$A=1$	$A=0$
$A=0$	$A=1$

Efficiency factors

	A	B	C	$A+B$	$A+C$	$B+C$	$A+B+C$
Rows	0	0	0	0	0	0.5	0.5
Columns	0	0	0	0.5	0.5	0	0
Rows.Columns	1	1	1	0.5	0.5	0.5	0.5

A 3^3 factorial experiment in two 9×9 squares

Characters	Square I			Square II		
Rows	$A+C$	$A+B+C$	$A+2B+C$	$B+C$	$A+B+C$	$A+2B+2C$
Columns	$A+2C$	$A+B+2C$	$A+2B+2C$	$B+2C$	$A+2B+C$	$A+B+2C$
Subsquares		B			A	

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Rows	$A+C$	$A+B+C$	$A+2B+C$	$B+C$	$A+B+C$	$A+2B+2C$
Columns	$A+2C$	$A+B+2C$	$A+2B+2C$	$B+2C$	$A+2B+C$	$A+B+2C$
Subsquares	B			A		

Efficiency factors

	main effects	A-by-B	A-by-C	B-by-C	A-by-B-by-C
Rows.Columns	1	1	$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{3}$

Quasi-Latin rectangles

A quasi-Latin rectangle is a row-column design with

- k rows and l columns, where $k \neq l$;
- v treatments, each replicated r times, so that $vr = kl$;
- at least one of k, l is less than v.

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A quasi-Latin rectangle is a row-column design with

- k rows and l columns, where $k \neq l$;
- v treatments, each replicated r times, so that $vr = kl$;
- at least one of k, l is less than v.

We consider the special case where

- $v = p^m$ for some prime p, and
- at least one of k, l divides v.

Case 1: k and l both divide v

This is similar to the previous construction.

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A 2^4 factorial experiment in a 4×8 rectangle.

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A 2^4 factorial experiment in a 4×8 rectangle.

confound $A+B+C+D$								
confound $A+C$								
	confound $A+B+C$, $A+B+D$ and $C+D$				confound $A+C+D$, $B+C+D$ and $A+B$			

This is similar to the previous construction.

A 2^4 factorial experiment in a 4×8 rectangle.

confound $A+B+C+D$								
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The character to confound with sub-rectangles must not be

- a row character

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confound $A+C$								
	confound $A+B+C$, $A+B+D$ and $C+D$				confound $A+C+D$, $B+C+D$ and $A+B$			

The character to confound with sub-rectangles must not be

- a row character
- a column character

Case 1: k and l both divide v

This is similar to the previous construction.

A 2^4 factorial experiment in a 4×8 rectangle.

confound $A+B+C+D$								
confound $A+C$								
	confound $A+B+C$, $A+B+D$ and $C+D$				confound $A+C+D$, $B+C+D$ and $A+B$			

The character to confound with sub-rectangles must not be

- a row character
- a column character
- the sum of a row character and a column character.

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A 2^4 factorial experiment in a 4×8 rectangle.

<table>
<tr><td rowspan="2">confound
$A+B+C+D$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">confound
$A+C$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td colspan="4">confound $A+B+C$,
$A+B+D$ and $C+D$</td><td colspan="4">confound $A+C+D$,
$B+C+D$ and $A+B$</td></tr>
</table>

The character to confound with sub-rectangles must not be

- a row character
- a column character
- the sum of a row character and a column character.

The unique possibility is $B+D$.

Case 2: k divides v, and v divides l

To avoid the difficulties caused by restricted choice of sub-rectangle character:

1. use column characters as before;

Case 2: k divides v, and v divides l

To avoid the difficulties caused by restricted choice of sub-rectangle character:

1. use column characters as before;
2. the number of columns is a multiple of v,
   so use the algorithm from Hall's Marriage Theorem
   to re-arrange the treatments in each column
   so that each row consists of complete replicates.

A 2^3 experiment in a 4×8 rectangle

confound $A+B+C$		confound $A+B$		confound $A+C$		confound $B+C$	
$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(0,1,0)$
$(1,1,0)$	$(0,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,0)$	$(1,1,0)$	$(1,0,0)$	$(1,1,0)$
$(1,0,1)$	$(0,0,1)$	$(1,1,0)$	$(0,1,0)$	$(1,0,1)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$
$0,1,1)$	$(1,1,1)$	$(1,1,1)$	$(0,1,1)$	$(1,1,1)$	$(0,1,1)$	$(1,1,1)$	$(1,0,1)$

A 2^3 experiment in a 4×8 rectangle

confound $A+B+C$		confound $A+B$		confound $A+C$		confound $B+C$	
$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$(0,1,0)$
$(1,1,0)$	$(0,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,0)$	$(1,1,0)$	$(1,0,0)$	$(1,1,0)$
$(1,0,1)$	$(0,0,1)$	$(1,1,0)$	$(0,1,0)$	$(1,0,1)$	$(0,0,1)$	$(0,1,1)$	$(0,0,1)$
$0,1,1)$	$(1,1,1)$	$(1,1,1)$	$(0,1,1)$	$(1,1,1)$	$(0,1,1)$	$(1,1,1)$	$(1,0,1)$

Re-arrange treatments in each column to make each row a complete replicate:

$(0,0,0)$	$(1,0,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,0)$	$(0,1,1)$	$(1,1,1)$	$(1,1,0)$
$(1,1,0)$	$(1,1,1)$	$(0,0,0)$	$(1,0,0)$	$(1,0,1)$	$(0,0,1)$	$(0,1,1)$	$(0,1,0)$
$(1,0,1)$	$(0,1,0)$	$(1,1,1)$	$(0,1,1)$	$(0,0,0)$	$(1,1,0)$	$(1,0,0)$	$(0,0,1)$
$(0,1,1)$	$(0,0,1)$	$(1,1,0)$	$(0,1,0)$	$(1,1,1)$	$(1,0,0)$	$(0,0,0)$	$(1,0,1)$

Case 3: other

A 3^3 factorial experiment in a 12×9 rectangle.
Divide the rectangle into boxes whose size is a power of 3.

3 rows									
3 rows									
3 rows									
3 rows									

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A 3^3 factorial experiment in a 12×9 rectangle.
Divide the rectangle into boxes whose size is a power of 3.

3 rows									
3 rows									
3 rows									
3 rows									

1. Confound $A+B+C$, $A+2B$, $2A+C$ and $2B+C$ to get 9 "groups" of size 3 (the same size as the boxes).

Case 3: other

A 3^3 factorial experiment in a 12×9 rectangle.
Divide the rectangle into boxes whose size is a power of 3.

3 rows	5	6	4	8	9	7	2	3	1
3 rows	6	4	5	9	7	8	3	1	2
3 rows	8	9	7	2	3	1	5	6	4
3 rows	9	7	8	3	1	2	6	4	5

1. Confound $A+B+C$, $A+2B$, $2A+C$ and $2B+C$ to get 9 "groups" of size 3 (the same size as the boxes).
2. Arrange groups 1–9 in a good 4×9 row–column design.

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3 rows	5	6	4	8	9	7	2	3	1
3 rows	6	4	5	9	7	8	3	1	2
3 rows	8	9	7	2	3	1	5	6	4
3 rows	9	7	8	3	1	2	6	4	5

1. Confound $A+B+C$, $A+2B$, $2A+C$ and $2B+C$ to get 9 "groups" of size 3 (the same size as the boxes).
2. Arrange groups 1–9 in a good 4×9 row–column design.
3. Confound other characters in each set of 3 rows: for example, $A+B+2C$, $A+2B+C$, $2A+B+C$, $B+C$.