Circular designs balanced for neighbours at distances one and two

R. A. Bailey

Queen Mary University of London

r.a.bailey@qmul.ac.uk

Ongoing joint work with Tank Aldred (University of Otago, New Zealand), Brendan McKay (ANU, Australia) and Ian Wanless (Monash University, Australia)

Abstract

We consider experiments where the experimental units are arranged in a circle or in a single line in space or time. If neighbouring treatments may affect the response on an experimental unit, then we need a model which includes the effects of direct treatments, left neighbours and right neighbours. It is desirable that each ordered pair of treatments occurs just once as neighbours and just once with a single unit in between. A circular design with this property is equivalent to a special type of quasigroup.

In one variant of this, self-neighbours are forbidden. In a further variant, it is assumed that the left-neighbour effect is the same as the right-neighbour effect, so all that is needed is that each unordered pair of treatments occurs just once as neighbours and just once with a single unit in between.

I shall report progress on finding methods of constructing the three types of design.

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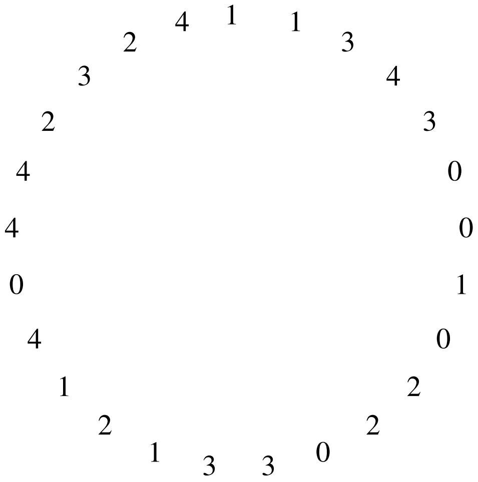
The marine biologist required that

(i) each ordered pair of items should occur just once as ordered neighbours around the circumference of the tank;

(ii) each ordered pair of items should occur just once with a single item in between them, in order.

A circular design for 5 treatments with neighbour balance at distances one and two



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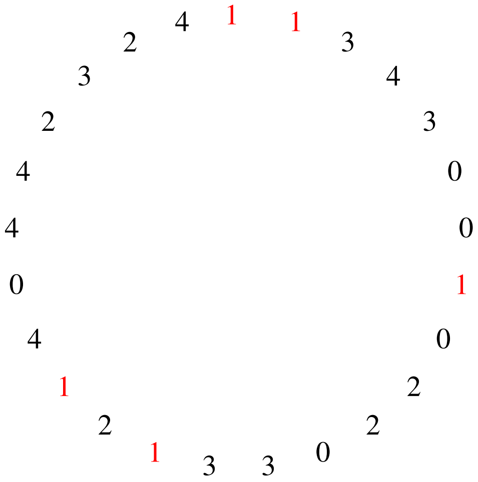

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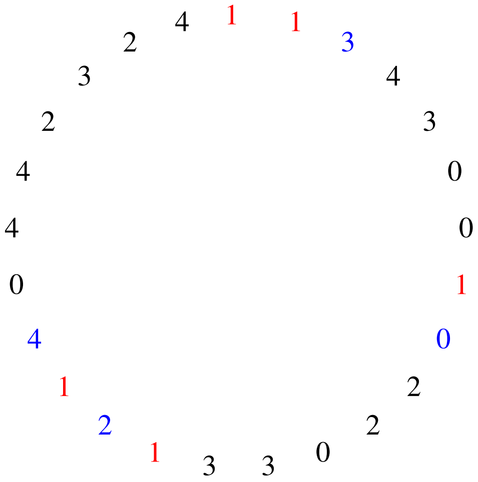

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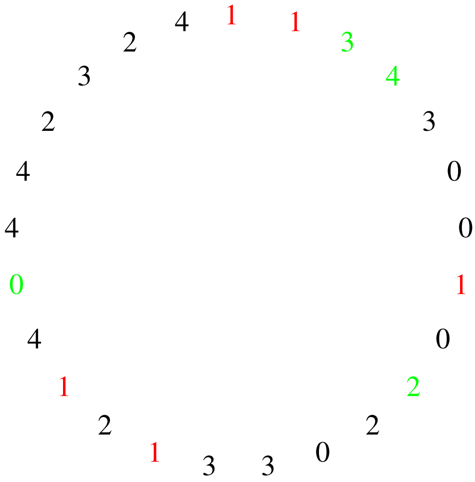
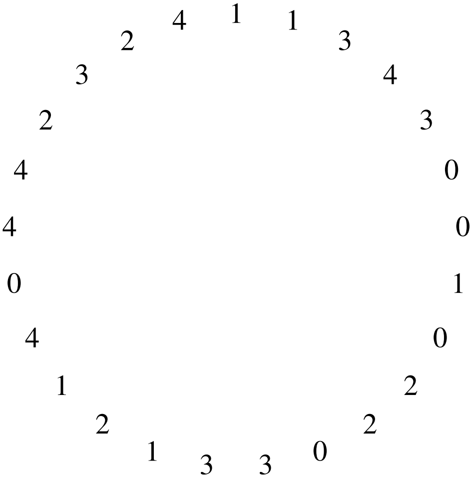


The lazy way to write the design

(1 1 3 4 3 0 0 1 0 2 2 0 3 3 1 2 1 4 0 4 4 2 3 2 4)



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The direct treatment effects δ,
the left neighbour effects λ
and the right neighbour effects ρ
can be estimated orthogonally of each other
in a experiment of this size
if and only if the design has neighbour balance at distances one and two.

Those conditions again

Among the triples of the form

$$\big(\tau(i-1), \tau(i), \tau(i+1)\big),$$

each ordered pair of treatments occurs once in positions 1 and 2, once in positions 1 and 3, and once in positions 2 and 3.

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These are conditions for a Latin square whose rows and columns have the same labels as the symbols—a quasigroup.

Building the design from a quasigroup (Latin square)

The quasigroup operation $\circ$ is defined by

$$a \circ b = \text{symbol in row } a \text{ and column } b \text{ of the Latin square.}$$

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We can start with any ordered pair (x, y) and successively build the circular design from the quasigroup as

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

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This quasigroup gives a design with four separate circles, not one.

$$(\; A \quad A \quad B \quad A \quad C \quad D \;)$$

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Email from Ian Wanless on 11 July 2010:

> *Back in Australia now and awake in the middle of the night... but wanted to let you know that in my sleeplessness I've solved that parity question.*

Variant I: no self-neighbours

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Preece (1976) showed that, for overall balance, the missing pairs at distance two must also be the self-pairs.

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Our circular design is equivalent to an idempotent quasigroup in which the $n(n-1)$ off-diagonal cells give a single circle.

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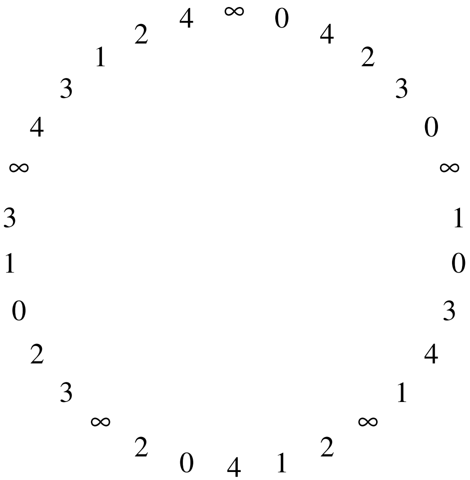
Differences at distance one come from the original sequence; difference at distance two are the neighbour sums.

A circular design for 6 treatments with no self-neighbours at distance one or two



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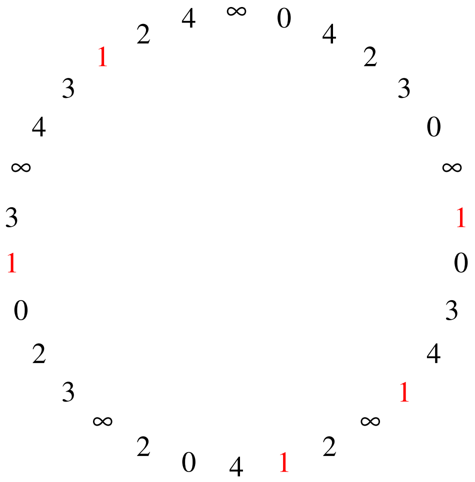

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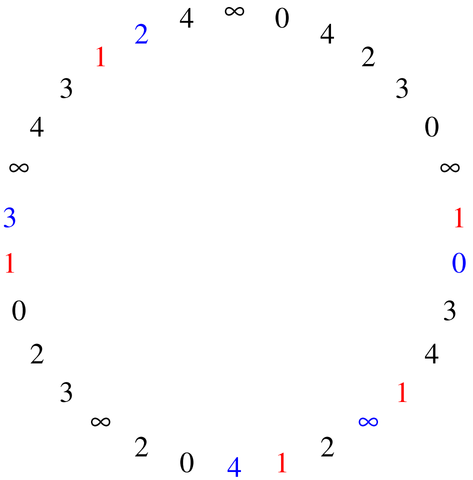

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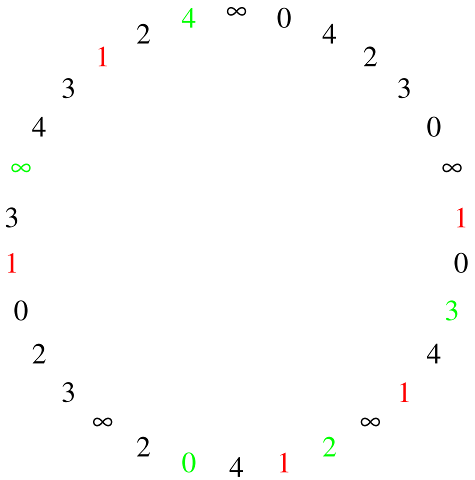

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Solution for variant I

Theorem

Given an initial sequence of the non-zero integers modulo $n-1$ satisfying those conditions, that construction always produces an idempotent Eulerian circular sequence.

Theorem

Such an initial sequence can be constructed whenever $n \geq 6$.

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Any triple (a,b,a) gives b as a neighbour of a on both sides, so there can be no such triples.

Model and variance

$$Y_i = \lambda_{\tau(i-1)} + \delta_{\tau(i)} + \lambda_{\tau(i+1)} + \varepsilon_i,$$

In vector form

$$\mathbf{Y} = X_1\delta + X_2\lambda + \varepsilon.$$

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The neighbour conditions in the design imply that each treatment occurs r times, where $r = (n-1)/2$.
Also $X_1^\top X_1 = rI$, $X_1^\top X_2 = J - I$ and $X_2^\top X_2 = 2rI + (J - I)$,
where I is the $n \times n$ identity matrix and J is the $n \times n$ all-1 matrix.

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Some calculations show that the variance of the estimator of the difference between two direct effects is

$$\frac{2(2r-1)}{(r-1)(2r+1)}\sigma^2$$

while that for the difference between two neighbour effects is

$$\frac{2r}{(r-1)(2r+1)}\sigma^2.$$

Construction when $n = 9$

The treatments are the integers modulo 9.

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$$(1\ 3\ 8\ 2\ 3\ 5\ 1\ 4\ 5\ 7\ 3\ 6\ 7\ 0\ 5\ 8\ 0\ 2\ 7\ 1\ 2\ 4\ 0\ 3\ 4\ 6\ 2\ 5\ 6\ 8\ 4\ 7\ 8\ 1\ 6\ 0)$$

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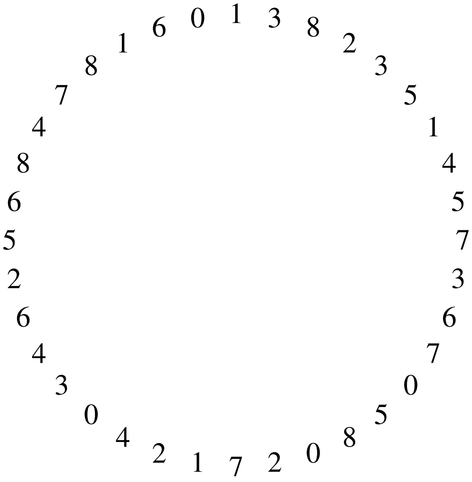
Differences at distance one come from the original sequence; difference at distance two are the neighbour sums.

A circular design for 9 treatments with undirectional neighbour balance at distances one and two



(1 3 8 2 3 5 1 4 5 7 3 6 7 0 5 8 0 2 7 1 2 4 0 3 4 6 2 5 6 8 4 7 8 1 6 0)

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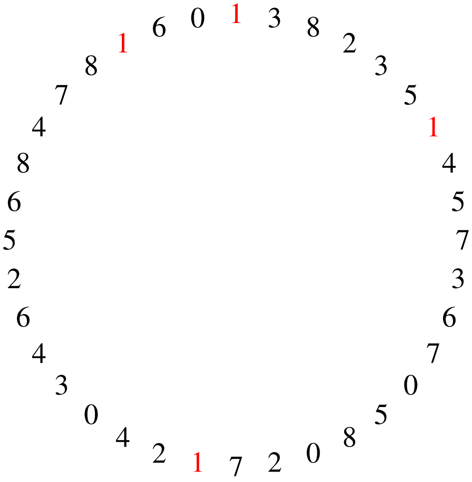

(1 3 8 2 3 5 1 4 5 7 3 6 7 0 5 8 0 2 7 1 2 4 0 3 4 6 2 5 6 8 4 7 8 1 6 0)

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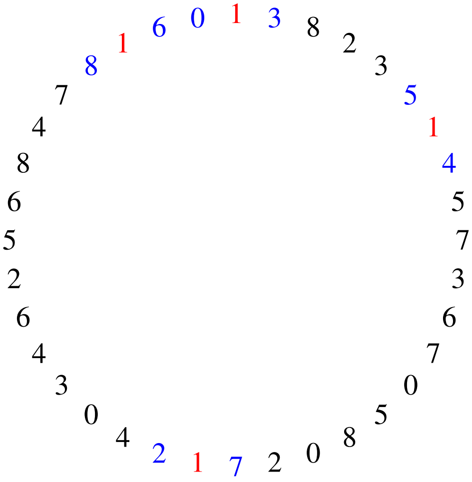

(1 3 8 2 3 5 1 4 5 7 3 6 7 0 5 8 0 2 7 1 2 4 0 3 4 6 2 5 6 8 4 7 8 1 6 0)

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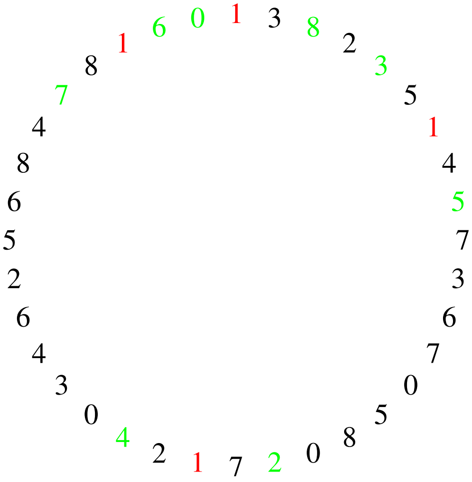

(1 3 8 2 3 5 1 4 5 7 3 6 7 0 5 8 0 2 7 1 2 4 0 3 4 6 2 5 6 8 4 7 8 1 6 0)

Solution for variant II

Theorem

Given an initial circular sequence of $(n-1)/2$ of the integers modulo n satisfying those conditions, that construction always produces a circular sequence balanced for undirected neighbours at distances one and two.

Theorem

Such an initial sequence can be constructed whenever n is odd and $n \geq 9$. There is also such a circular sequence when $n = 7$.