Experiments in rectangular areas: restricted randomization or row-column designs?

R. A. Bailey

r.a.bailey@qmul.ac.uk

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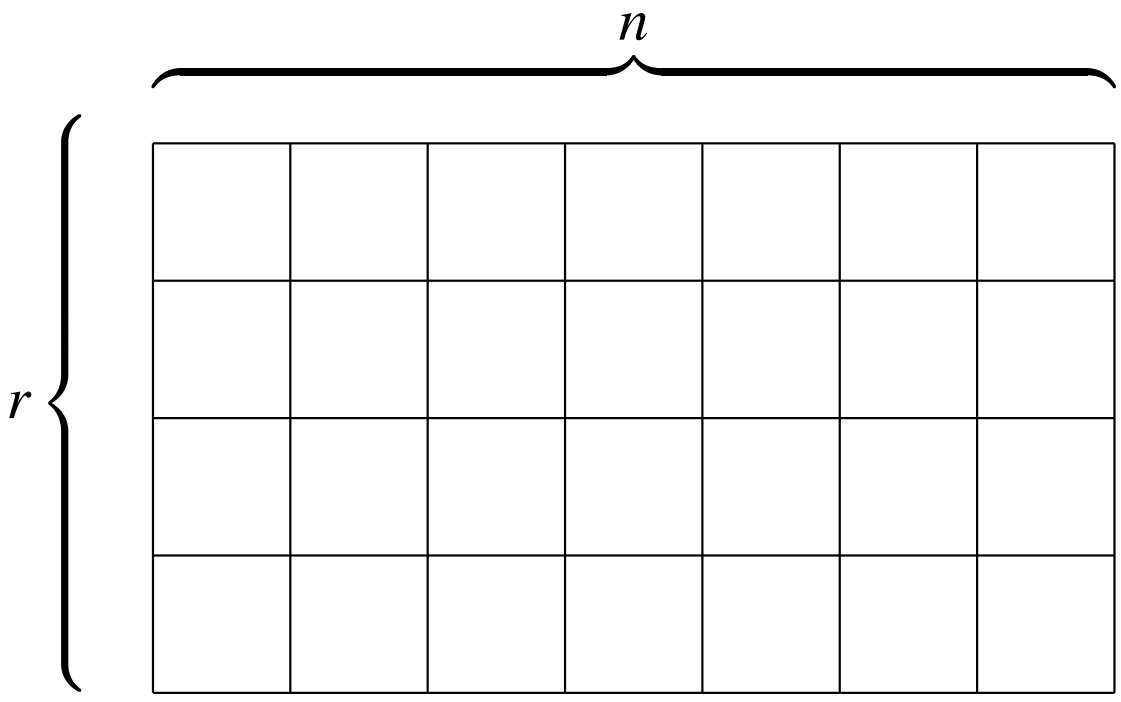
The problem

An agricultural experiment to compare n treatments.
The experimental area has r rows and n columns.



Use a randomized complete-block design with rows as blocks.
(In each row, choose one of the $n!$ orders with equal probability.)

What should we do if the randomization produces a plan with one treatment always at one side of the rectangle?

Example

Federer (1955 book): guayule trees

B	D	G	A	F	C	E
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B	A	C	F	G	E	D
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Deliberately construct a design in which no treatment occurs more than once in any column.

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Solution: Use a carefully chosen Latinized design; REML/ANOVA estimates of variance components

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  but the estimated variance is unbiased
  when averaged over all comparisons and
  all possible randomized plans.

Assumed model

Y_α is the response on plot α.

$E(Y_\alpha) = \theta_i$ where i is the treatment on α.

$$\begin{aligned} \mathrm{Var}(Y_\alpha) &= \sigma^2 \qquad \text{for all } \alpha \\ \mathrm{Cov}(Y_\alpha, Y_\beta) &= \begin{cases} \rho\sigma^2 & \text{if } \alpha \neq \beta \text{ in same row} \\ \tau\sigma^2 & \text{if } \alpha \neq \beta \text{ in same column} \\ 0 & \text{if } \alpha \neq \beta \text{ otherwise} \end{cases} \end{aligned}$$

with $0 \leq \rho \leq 1$ and $0 \leq \tau \leq 1$.

Concurrence

$\lambda_{ij} =$ number of pairs of plots in the same column getting treatments i and j.

B	D	G	A	F	C	E
A	G	C	D	F	B	E
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$$\lambda_{AD} = 0+1+0+1+0+0+1 = 3$$

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$$\begin{aligned}\lambda_{AD} &= 0+1+0+1+0+0+1 = 3\\ \lambda_{AB} &= 2+1+0+0+0+1+0 = 4\end{aligned}$$

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$$\begin{aligned}
\lambda_{AD} &= 0+1+0+1+0+0+1 = 3 \\
\lambda_{AB} &= 2+1+0+0+0+1+0 = 4 \\
\lambda_{AA} &= 1+1+0+1+0+1+1 = 5
\end{aligned}$$

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\lambda_{AA} &= 1+1+0+1+0+1+1 = 5 \\
\lambda_{BB} &= 4+1+0+0+1+1+0 = 7
\end{aligned}$$

Pairwise variance

$$
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\end{aligned}
$$

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\begin{aligned}
V_{ij} &= \text{variance of the estimator of } \theta_i - \theta_j \\
&= \frac{\sigma^2}{r^2}\left[2r - 2r\rho + (\lambda_{ii} - r)\tau + (\lambda_{jj} - r)\tau - 2\lambda_{ij}\tau\right]
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same plot

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$\nwarrow$ same row

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$$= \frac{\sigma^2}{r^2}\left[2r(1-\rho) + (\lambda_{ii} + \lambda_{jj} - 2\lambda_{ij} - 2r)\tau\right]$$

Pairwise variance in the example

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From $V_{BG} = \frac{2\sigma^2}{5}\left[1 - \rho - \frac{4}{5}\tau\right]$

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From $V_{BG} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{4}{5}\tau\right]$ to $V_{EF} = \frac{2\sigma^2}{5}\left[1-\rho+\tau\right]$

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$$\text{From} \quad V_{BG} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{4}{5}\tau\right] \quad \text{to} \quad V_{EF} = \frac{2\sigma^2}{5}\left[1-\rho+\tau\right]$$

$$\text{with average} \quad V = \frac{2\sigma^2}{5}\left[1-\rho-\frac{1}{15}\tau\right].$$

Continue to randomize and analyse as usual: summary

- Simple to construct.
- Simple to randomize.
- Simple to analyse.
- Some treatment comparisons in some experiments will have a specially low or specially high variance,
  but the estimated variance is unbiased
  when averaged over all comparisons and
  all possible randomized plans.

Simple restricted randomization

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- Inefficient to produce plans: many will have to be rejected.
  For the 5×7 rectangle, the proportion of plans with no repeat in any column is only 0.000006.
- The actual variance of treatment comparisons is lower,
  but the estimate of that variance is higher.

Calculations

$\lambda_{ij} =$ number of pairs of plots in the same column getting treatments i and j

$$\text{Note that} \quad \sum_{j=1}^{n} \lambda_{ij} = r^2 \quad \text{for each } i.$$

$$\text{We know that} \quad V_{ij} = \frac{\sigma^2}{r^2}\left[2r(1-\rho) + (\lambda_{ii} + \lambda_{jj} - 2\lambda_{ij} - 2r)\tau\right]$$

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$$\text{Put} \quad V = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} V_{ij} \quad \text{and put} \quad D = \sum_{i=1}^{n} \lambda_{ii}.$$

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$$\text{Calculations give} \quad V = \frac{2\sigma^2}{r^2}\left[r(1-\rho) + \left(\frac{D - r^2}{n-1} - r\right)\tau\right]$$

Spectral form of covariance matrix

$$\mathrm{Cov}(\mathbf{Y}) = \sigma^2[\mathbf{I} + \rho(\mathbf{R} - \mathbf{I}) + \tau(\mathbf{C} - \mathbf{I})]$$

where $\mathbf{I}$ is the identity matrix,
$\mathbf{R}$ is the matrix whose (α, β)-entry is equal to 1 if plots α and β are in the same row and to 0 otherwise,
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$$\text{So} \quad \mathrm{Cov}(\mathbf{Y}) = \xi_0\mathbf{S}_0 + \xi_1\mathbf{S}_1 + \xi_2\mathbf{S}_2 + \xi_3\mathbf{S}_3, \quad \text{where}$$

$$\begin{aligned} \xi_0 &= \sigma^2(1+(n-1)\rho+(r-1)\tau) \\ \xi_1 &= \sigma^2(1-\tau+(n-1)\rho) \\ \xi_2 &= \sigma^2(1-\rho+(r-1)\tau) \\ \xi_3 &= \sigma^2(1-\rho-\tau) \end{aligned}$$

rows stratum $\mathbf{S_1} = \frac{1}{n}\mathbf{R} - \frac{1}{rn}\mathbf{J}$ $\mathbf{S_2} = \frac{1}{r}\mathbf{C} - \frac{1}{rn}\mathbf{J}$ columns stratum

grand mean $\mathbf{S}_0 = \frac{1}{rn}\mathbf{J}$ $\mathbf{S}_3 = \mathbf{I} - \frac{1}{n}\mathbf{R} - \frac{1}{r}\mathbf{C} + \frac{1}{rn}\mathbf{J}$ plots stratum

Strata for analysis of variance

stratum	df	variance
mean	1	ξ_0
rows	$r-1$	ξ_1
columns	$n-1$	ξ_2
plots	$(r-1)(n-1)$	ξ_3

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$$E(\text{SS for contrasts} \perp \text{to rows}) = Q + (n-1)\xi_2 + (n-1)(r-1)\xi_3$$

$$=$$

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$$\begin{aligned} E(\text{SS for contrasts} \perp \text{to rows}) &= Q + (n-1)\xi_2 + (n-1)(r-1)\xi_3 \\ &= Q + (n-1)(y+r-1)\xi_3. \quad (y = \xi_2/\xi_3) \end{aligned}$$

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$$\text{So} \quad E(\text{MS residual}) = E\left(\frac{\text{SS residual}}{(n-1)(r-1)}\right) = \frac{1}{r-1}\left[(y+r-1)\xi_3 - \frac{rV}{2}\right].$$

Overestimation of variance

The estimator of V is $\hat{V} = \dfrac{2M}{r}$, where $M = \text{MS residual}$.

We have shown that $E\left(\dfrac{2M}{r}\right) = \dfrac{2}{r-1}\left[\dfrac{(y+r-1)}{r}\xi_3 - \dfrac{V}{2}\right]$,

so smaller $V \Longrightarrow$ larger $\hat{V}$.

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Reparametrizing: $V = \dfrac{2\xi_3}{r}\left[1 + \dfrac{(y-1)(D-r^2)}{r^2(n-1)}\right]$ with $D = \sum_{i=1}^{n} \lambda_{ii}$.

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$$\begin{aligned} V &= \frac{2\xi_3}{r}\left[1 + \frac{(y-1)(n-r)}{r(n-1)}\right] \\ \text{and} \quad \frac{2E(M)}{r} &= \frac{2\xi_3}{r}\left[1 + \frac{(y-1)n}{r(n-1)}\right] \end{aligned}$$

which over-estimates V by $2(y-1)\xi_3/[r(n-1)]$.

Simple restricted randomization: summary

Keep re-randomizing until you get a plan with no treatment more than once in any column. Analyse as usual.

- Inefficient to produce plans: many will have to be rejected.
- Variance is overestimated:

$$V = \frac{2\xi_3}{r}\left[1 + \frac{(y-1)(n-r)}{r(n-1)}\right]$$

and

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- Genuine treatment differences may not be detected.

Use a Latinized design, but analyse as usual

Deliberately construct a design in which no treatment occurs more than once in any column.
Easy to do this directly, eg

A						
C						
G						
B						
D						

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D	E	F				

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D	E	F	G	A	B	C

Randomize rows, columns, treatments.

Use a Latinized design, but analyse as usual

Deliberately construct a design in which no treatment occurs more than once in any column.
Easy to do this directly, eg

A	B	C	D	E	F	G
C	D	E	F	G	A	B
G	A	B	C	D	E	F
B	C	D	E	F	G	A
D	E	F	G	A	B	C

Randomize rows, columns, treatments.

Same bias in estimator of variance as for simple restricted randomization.

Super-valid restricted randomization

- Needs tables of designs.

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- The estimator of variance is unbiased
  when averaged over all comparisons in this one experiment.

Condition for unbiased estimator of variance

$$\text{We have} \quad V = \frac{2\xi_3}{r}\left[1 + \frac{(y-1)(D-r^2)}{r^2(n-1)}\right]$$

$$\text{with} \quad D = \sum_{i=1}^{n} \lambda_{ii},$$

$$\text{and} \quad E\left(\hat{V}\right) = \frac{2}{r-1}\left[\frac{(y+r-1)}{r}\xi_3 - \frac{V}{2}\right],$$

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If each pair of rows has one column with the same treatment but no treatment occurs more than twice in any column then $D = r(r+n-1)$ and $V = E(\hat{V})$.

A design from the tables

A	B	C	D	E	F	G
D	E	F	C	A	B	G
A	G	F	B	C	E	D
D	B	G	F	C	A	E
G	E	C	B	D	A	F

1. In every pair of rows, there is exactly one column in which the two treatments are the same.

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3. If $m_i =$ the number of columns in which treatment i occurs twice, then $m_i - m_j \in \{-1, 0, 1\}$ for all other treatments j.
4. Subject to conditions (1)–(3), the spread of the variances of the estimators of simple treatment differences is as small as possible.

Pairwise variances in the example

A	B	C	D	E	F	G
D	E	F	C	A	B	G
A	G	F	B	C	E	D
D	B	G	F	C	A	E
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$$\text{Minimum} \quad V_{AD} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{2}{5}\tau\right]$$

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one layout, normal method

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Average	$V = \frac{2\sigma^2}{5}(1-\rho)$	$\cdots -\frac{1}{15}\tau$	$\cdots -\frac{2}{3}\tau$	
		one layout, normal method	simple restricted randomization	

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- There is no separate estimate of ρ or τ (or y),
  so treatments must be randomized
  and a single standard error given for all differences.

Efficient row-column designs

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Columns form an Incomplete-block design (IBD)

Given an incomplete-block design for
n treatments in n blocks of size r,
define the number A $(0 < A < 1)$, depending on the design, by

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Randomize rows and columns.

IBD continued

Analyse by fitting rows, columns and treatments.

$$E(\text{MS residual}) = \xi_3$$

$$V_{ij} = \frac{2\xi_3}{rA_{ij}} \qquad \text{where } A_{ij} \text{ is known}$$

$$V = \frac{2\xi_3}{rA} = \frac{2\sigma^2}{rA}(1-\rho-\tau)$$

Example of a row-column design

A	B	C	D	E	F	G
B	C	D	E	F	G	A
C	D	E	F	G	A	B
D	E	F	G	A	B	C
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$$
\begin{aligned}
V_{AB} &= 1.044 \times \frac{2}{5}\xi_3 \\
V_{AC} &= 1.089 \times \frac{2}{5}\xi_3 \\
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V &= 1.075 \times \frac{2}{5}\xi_3
\end{aligned}
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$$V_{AB} = 1.044 \times \frac{2}{5}\xi_3 \qquad \text{normal method}$$

$$V_{AC} = 1.089 \times \frac{2}{5}\xi_3 \qquad V = \frac{2}{5}\left(\frac{y+4}{5}\right)\xi_3$$

$$V_{AD} = 1.091 \times \frac{2}{5}\xi_3 \qquad \text{averaged over randomizations}$$

$$V = 1.075 \times \frac{2}{5}\xi_3 \qquad \text{N.B. } y = \xi_2/\xi_3 \geq 1$$

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- Needs tables of designs.
- Randomize rows and columns.
- More complicated analysis
  (should be available in software).
- Average variance may be less than, or more than,
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- Unbiased estimator of the variance of every treatment contrast.
- There is no need to randomize treatments; the most important differences can be given the lowest variance.

Comparing super-valid restricted randomization and efficient row-column designs

$$y = \frac{\xi_2}{\xi_3} = \frac{\text{columns stratum variance}}{\text{plots stratum variance}} \geq 1 \quad \text{(we believe)}$$

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	n					
r	5	6	7	8	9	10
3	1.68	1.83	1.86	2.02	2.13	2.27
4	1.26	1.47	1.57	1.71	1.80	1.86
5		1.21	1.37	1.48	1.58	1.64

Use a carefully chosen Latinized design with REML/ANOVA estimates of variance components

Choose a design with the λ_{ij} as equal as possible.
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Hence unbiased estimators of ξ_2 and ξ_3 and of

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But this estimator of V does not have a χ^2 distribution,
so how do we do hypothesis tests?
Also, there are so few effective df for ξ_2 that
these estimates have very poor precision.