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Resolved designs viewed as sets of partitions

1 Incomplete-block Designs

In this paper I take the view, common among statisticians, that a design is an allocation of a set Γ of n treatments to a set Ω of plots: see [2]. (The word *plot* does not necessarily denote a piece of ground: it is used for any type of experimental unit.) Thus the design may be thought of as a function f from Ω onto Γ : plot ω receives treatment γ if $f(\omega) = \gamma$. If Ω is partitioned into *blocks* then f is called a *block design*. A block design is *proper* if all its blocks have the same size. In this paper all block designs are proper, having b blocks of size k .

Suppose that the restriction of f to each block is injective, that is,

$$\text{there are no two distinct plots } \psi \text{ and } \omega \text{ such that } \psi \text{ and } \omega \text{ are in the} \quad (1.1) \\ \text{same block and } f(\psi) = f(\omega).$$

Then $k \leq n$. If $k = n$ the design is called a *complete-block design*; if $k < n$ it is an *incomplete-block design*.

The names of the blocks are immaterial, as is the order of plots within any block: these are both taken care of by a process called *randomization*. Therefore it is legitimate to identify an incomplete-block design f with a family of k -subsets of Γ —the set $\{f(\omega) : \omega \in \Phi\}$ for each block Φ . Many of the designs in this paper are illustrated using this identification. Furthermore, we say that treatment γ *occurs* in block Φ if there is an ω in Φ for which $f(\omega) = \gamma$; and that treatments γ and δ *concur* in block Φ if there are plots ψ and ω in Φ such that $f(\psi) = \gamma$ and $f(\omega) = \delta$.

Figures 1.1 and 1.2 show two incomplete-block designs. In both designs all the treatments occur in the same number of blocks. Such a design is called *equireplicate*. The design in Figure 1.1 also has the same number of blocks as treatments: such a design is called *symmetric* (traditionally) or (with more justification) *square*.

The two most important statistical properties of incomplete-block designs are *balance* and *resolvability*. A design is *balanced* if every pair of distinct treatments concurs in the same number of blocks. The design in Figure 1.1 is balanced; that in Figure 1.2 is not. Balance implies that the design is equireplicate.

A design is *resolvable* if the set of blocks can be partitioned into superblocks each of which is complete in the sense that all treatments occur in it once. In a resolvable design all treatments occur r times, where r is the number of superblocks; and k divides n , because n/k is the number of blocks per superblock.

In a real experiment, the partition of the blocks into superblocks is usually made *before* the design f is chosen. For example, if k trials can be run per day, then days might be blocks and weeks superblocks. So I prefer to call a design *resolved* if

0 1 4 6	1 2 5 7	2 3 6 8	3 4 7 9	4 5 8 X
5 6 9 E	6 7 X T	7 8 E 0	8 9 T 1	9 X 0 2
	X E 1 3	E T 2 4	T 0 3 5	

Figure 1.1 A balanced incomplete-block design for 13 treatments in 13 blocks of size 4

the already-given superblocks are complete. (I am grateful to D. A. Preece for this suggestion.) The design in Figure 1.2 is resolved.

(In the statistical literature a superblock is often called a *replicate*. I find this confusing. For one thing, the design in Figure 1.1 is often said to have four replicates even though it has no superblocks: this just means that every treatment occurs four times. More importantly, whether or not a particular n -subset of Ω is a replicate depends on the allocation of treatments, while the superblocks, if any, exist irrespective of treatment allocation.)

2 Partitions from Incomplete-block Designs

Incomplete-block designs give rise to families of partitions in two distinct ways.

Let B be the partition of Ω into blocks. We may identify f with the partition of Ω into the sets $f^{-1}(\gamma)$ for γ in Γ . Thus the incomplete-block design defined by Ω , Γ and f may be regarded as the pair of partitions (B, f) of Ω so long as the names of the treatments are not important. Interchanging the rôles of the two partitions gives the *dual* design.

Let S be the partition of Ω into superblocks, if any. Then B is a refinement of S , and a resolved incomplete-block design may be regarded as the triple of partitions (S, B, f) of Ω .

0 1 2 3	0 1 4 8	2 4 X E
4 5 6 7	2 5 6 9	3 5 6 8
8 9 X E	3 7 X E	7 9 0 1

Figure 1.2 A resolved incomplete-block design for 12 treatments in 3 superblocks of 3 blocks of size 4

A resolved design also gives a family of partitions of Γ . Let the superblocks be $\Omega_1, \Omega_2, \dots, \Omega_r$, and let f_i be the restriction of f to Ω_i . Then f_i is a bijection from Ω_i to Γ . Applying f_i to the blocks contained in Ω_i gives a partition F_i of Γ into s classes of size k where $s = n/k$; the classes are $\{f_i(\omega) : \omega \in \Phi\}$ for blocks Φ contained in Ω_i .

(Statisticians call partitions such as B, S and f *factors* because they may explain different values of a response variable on different plots, irrespective of the design used. This term is usually confined to partitions with a physical meaning, and is not used for induced partitions such as F_1, \dots, F_r .)

For most of the rest of the paper I shall concentrate on the relationship between a resolved incomplete-block design and the corresponding family of partitions F_1, F_2, \dots, F_r of Γ . It is necessary to describe some general theory about partitions of a set: this is done in Sections 3, 6 and 9.

3 Orthogonality between Partitions

In the context of statistical design, orthogonality is the most important relation between a pair of partitions on the same set.

We need to discuss partitions on Γ and also partitions on Ω . To cover both, let Δ be a set of size M , and let F be a partition of Δ into classes of size k . For δ in Δ , write $F(\delta)$ for the class of F containing δ .

Associated with Δ is the M -dimensional real vector space \mathbb{R}^Δ consisting of real M -tuples indexed by Δ . The partition F defines a subspace V_F of \mathbb{R}^Δ : it consists of all those vectors v in \mathbb{R}^Δ which satisfy:

$$v_\delta = v_\varepsilon \quad \text{whenever } F(\delta) = F(\varepsilon).$$

The dimension of V_F is equal to the number of classes of F , which is M/k . Under the usual inner product on \mathbb{R}^Δ , the vector v is in the orthogonal complement V_F^\perp of V_F if and only if

$$\sum_{\delta \in \Phi} v_\delta = 0 \quad \text{for every class } \Phi \text{ of } F.$$

The partition F also defines a square matrix P_F in $\mathbb{R}^{\Delta \times \Delta}$. The (δ, ε) -entry of P_F is equal to $1/k$ if $F(\delta) = F(\varepsilon)$ and to 0 otherwise. It may be readily checked that

$$vP_F = v \quad \text{if } v \in V_F$$

while

$$vP_F = 0 \quad \text{if } v \in V_F^\perp.$$

Thus P_F is the matrix of orthogonal projection onto V_F .

The partition with a single class consisting of the whole of the set Δ gives the 1-dimensional subspace V_0 consisting of constant vectors and its projection matrix P_0 , all of whose entries are equal to $1/M$.

Following [56], I define two partitions F and G on Δ to be *orthogonal to each other* if $P_FP_G = P_GP_F$. In notation this is expressed as $F \perp G$. The context of

Section 2 gives an immediate example. On Ω , B is a refinement of S , so $V_S \subset V_B$. Therefore $P_B P_S = P_S P_B = P_S$ and so $B \perp S$.

I define F and G to be *strictly orthogonal to each other* if $P_F P_G = P_G P_F = P_0$. There does not seem to be any established notation for strict orthogonality: I propose using $F \perp G$. (Warning: some authors use *orthogonal* where I use *strictly orthogonal*.)

The definitions of V_F , P_F and orthogonality may be extended to partitions whose class sizes are unequal: see [5]. However, we need equal class sizes to obtain the following simple result.

Proposition 3.1 *If F and G are partitions of Δ into M/k_1 classes of size k_1 and M/k_2 classes of size k_2 respectively, then $F \perp G$ if and only if $|\Phi \cap \Psi| = k_1 k_2 / M$ for every class Φ of F and every class Ψ of G . \diamond*

Thus, on Ω , B is strictly orthogonal to f only if the design is a complete-block design (assuming that $k \leq n$). The condition that a design be resolved is equivalent to $S \perp f$ (assuming that all superblocks have size n).

What about partitions on Γ ? I shall write V_i and P_i for V_{F_i} and P_{F_i} . Each partition F_i consists of s classes of size k , so Proposition 3.1 shows that $F_i \perp F_j$ if and only if the intersection of each class of F_i with each class of F_j has size $k^2/n = n/s^2 = k/s$. In particular, strict orthogonality is not possible unless s divides k .

(See [5] for a more thorough review of the rôle of orthogonal partitions in designed experiments.)

4 Designs from Strictly Orthogonal Partitions

Yates introduced balanced incomplete-block designs into the design of experiments in [60]. The first non-balanced incomplete-block designs used for experiments were the designs now called *square lattices*, introduced by Yates in [61]. They are resolved designs in which $k = s$ and so $n = s^2$ and in which $F_i \perp F_j$ for all i, j with $1 \leq i < j \leq r$.

A square lattice design is constructed as follows. Write the elements of Γ as an $s \times s$ square array. Let $\Lambda_1, \dots, \Lambda_{r-2}$ be mutually orthogonal $s \times s$ Latin squares. The classes of F_1 are the rows of Γ ; the classes of F_2 are the columns of Γ ; while, for $i \geq 3$, the classes of F_i are the letters of Λ_{i-2} . (Square lattice designs are also known as *nets* [10, Chapter X].)

Example 4.1 Let $k = s = 4$, $n = 16$ and $r = 4$. Let $\Gamma = \{a, b, \dots, p\}$, arranged in the square array in Figure 4.1(a). Figure 4.1(b) shows two mutually orthogonal 4×4 Latin squares Λ_1 and Λ_2 . The resulting square lattice design is in Figure 4.1(c). \diamond

A compact way of showing partitions on the same set is to write the elements of the set as a row, with one row beneath it for each partition. For resolved designs, we write the elements of Γ as the top row. The i -th row below that shows F_i , using any s symbols. Positions γ and δ in this row have the same symbol if and only

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

α	β	γ	δ
γ	δ	α	β
δ	γ	β	α
β	α	δ	γ

(a) The array Γ (b) orthogonal Latin squares Λ_1 and Λ_2

a b c d	a e i m	a f k p	a g l n
e f g h	b f j n	b e l o	b h k m
i j k l	c g k o	c h i n	c e j p
m n o p	d h l p	d g j m	d f i o

(c) The resolved incomplete-block design

Figure 4.1 Square lattice design for 16 treatments in 4 superblocks of 4 blocks of size 4. See Example 4.1.

Γ	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
F_1	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
F_2	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
F_3	1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1
F_4	1	2	3	4	3	4	1	2	4	3	2	1	2	1	4	3

Figure 4.2 Compact form of the design in Figure 4.1(c)

if $F_i(\gamma) = F_i(\delta)$; that is, if and only if γ and δ occur in the same block of the i -th superblock Ω_i . The correspondence between the s symbols and the s blocks is immaterial; nor is it necessary to use the same symbols within different superblocks.

A compact form of the square lattice design in Figure 4.1(c) is shown in Figure 4.2. If the symbols in the rows for F_3 and F_4 were replaced by A, \dots, D and α, \dots, δ respectively the construction from the Latin squares in Figure 4.1(b) would be transparent.

This compact form will be familiar to some readers as an *orthogonal array of strength two* on Γ : see [53, Chapter 2]. The defining property of such an array is precisely that each pair of distinct partitions be strictly orthogonal.

As was observed in [6], every orthogonal array of strength two on Γ gives rise to a resolved incomplete-block design with treatment set Γ in which the partitions F_i are pairwise strictly orthogonal. An example with $n = 12$, $s = 2$, $k = 6$ and $r = 5$ is shown in Figure 4.3. By Proposition 3.1, in such a design s must divide k : the design is a square lattice design if and only if $s = k$.

Bose [11] defined a resolved design to be *affine* if $|f(\Phi_i) \cap f(\Phi_j)|$ is independent of the choice of block Φ_i in superblock Ω_i and block Φ_j in superblock Ω_j for $i \neq j$. His motivation was not orthogonal partitions but conditions on blocks similar to the

Γ	0	1	2	3	4	5	6	7	8	9	X	∞
F_1	2	2	1	2	2	2	1	1	1	2	1	1
F_2	1	2	2	1	2	2	2	1	1	1	2	1
F_3	2	1	2	2	1	2	2	2	1	1	1	1
F_4	1	2	1	2	2	1	2	2	2	1	1	1
F_5	1	1	2	1	2	2	1	2	2	2	1	1

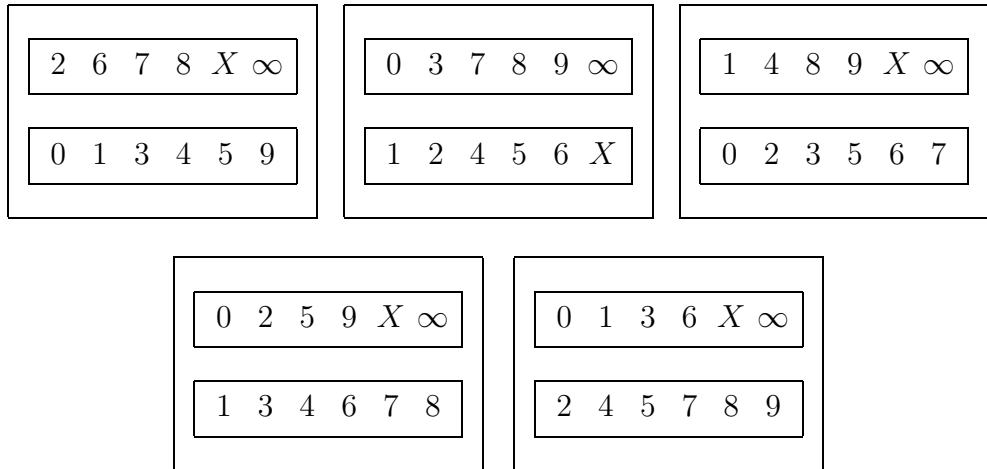


Figure 4.3 Orthogonal array of strength two and corresponding affine resolved incomplete-block design for 12 treatments in 5 superblocks of 2 blocks of size 6

condition on treatments that defines balance. However, Proposition 3.1 shows that:

Proposition 4.1 *A resolved incomplete-block design is affine resolved if and only if the partitions defined by its superblocks are pairwise strictly orthogonal.* \diamond

5 Optimality

What makes an incomplete-block design good?

The *concurrence matrix* C of a block design satisfying (1.1) is the matrix in $\mathbb{R}^{\Gamma \times \Gamma}$ whose (γ, δ) entry is equal to the *concurrence* of γ and δ , that is, the number of blocks in which γ and δ concur. Note that this means that the (γ, γ) diagonal entry is equal to the number of plots on which γ occurs. We consider here only the case in which all diagonal entries are equal to r ; that is, the design is equireplicate with *replication* r , whether or not it is resolved.

Put $L = I_{\Gamma} - (rk)^{-1}C$, where I_{Γ} is the identity matrix on Γ . As we shall see in Section 6, the eigenvalues of C lie in the interval $[0, rk]$, so those of L lie in the interval $[0, 1]$. The constant vectors are eigenvectors of L with eigenvalue 0. If this eigenvalue has multiplicity 1 then the design is said to be *connected*. Disconnected designs are used only in rather special circumstances (so-called *confounded* designs [17, Chapter 6] and *split-plot* designs [17, Chapter 7]) which we shall not consider in this paper.

Let the eigenvalues of L , with multiplicities, be $\lambda_1, \dots, \lambda_n$ with

$$1 \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0.$$

Of these, $\lambda_1, \dots, \lambda_{n-1}$ are called the *canonical efficiency factors* of the design. We want these to be as large as possible. However,

$$\sum_{i=1}^n \lambda_i = \text{tr } L = n(1 - k^{-1}),$$

so the average of $\lambda_1, \dots, \lambda_{n-1}$ is not a useful measure. The criterion A is defined to be the harmonic mean of the canonical efficiency factors, and to be zero if $\lambda_{n-1} = 0$. We call it the *overall efficiency factor* of the design.

The choice of the harmonic mean is not arbitrary. Let $\mathcal{V}_{\gamma\delta}$ be the variance of the estimator of the difference between the effect of γ and the effect of δ . If the variables observed on the plots in the experiment are independent with the same variance, then the average value of $\mathcal{V}_{\gamma\delta}$ (over all $\gamma \neq \delta$) is proportional to $1/A$: see [30, Chapter 2]. (If $A = 0$ then the design is disconnected and so there is at least one pair of treatments γ, δ for which the difference between their effects is not estimable. Conventionally we declare $\mathcal{V}_{\gamma\delta}$ to be infinite in this case.) Thus maximizing A minimizes the average variance.

Given two designs f and g for the same Ω , B and Γ , we distinguish their A criteria by writing A_f and A_g . Then f is *better* than g if $A_f > A_g$; while f is *optimal* if $A_f \geq A_g$ for all competing g .

The book [54] gives a very thorough account of optimality for incomplete-block designs. There are other optimality criteria: for example, the geometric mean of the canonical efficiency factors, and the minimum of the canonical efficiency factors, as well as some other functions of L that are not functions of the canonical efficiency factors. However, the A criterion is sufficient for this paper.

Denote by \mathcal{A}_Γ the subalgebra of $\mathbb{R}^{\Gamma \times \Gamma}$ spanned by the identity matrix I_Γ and the all-1 square matrix $J_{\Gamma, \Gamma}$. Then f is balanced if and only if $C \in \mathcal{A}_\Gamma$. Also, $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$ if and only if $C \in \mathcal{A}_\Gamma$. Finally, the harmonic mean of positive quantities is less than or equal to their arithmetic mean, with equality if and only if the quantities are all equal. This proves:

Theorem 5.1 (Kshirsagar [36]) *Balanced incomplete-block designs are optimal among incomplete-block designs.* \diamond

Note that we always assume that the competing designs have the same plot set Ω and block partition B . In this paper I also assume that they are all equireplicate. Sometimes we restrict the competing designs to be resolved: this may or may not change which designs are optimal.

Theorem 5.2 *Square lattice designs are optimal among incomplete-block designs, whether or not these are restricted to be resolved.* \diamond

Patterson and Williams proved that square lattice designs are optimal among resolved designs in [38]. Cheng and Bailey extended this to Theorem 5.2 in [16]. Yates showed good judgement when he invented square lattice designs 20 years before Kiefer [33, 34] defined optimality.

Theorem 5.3 ([38, 6]) *Affine resolved designs are optimal among resolved incomplete-block designs.* \diamond

Although Bose [11] was chiefly interested in balanced affine resolved designs, such as affine planes and Hadamard designs [14, Chapter 1], Theorem 5.3 again shows the good judgement of one of the giants in design of experiments.

Theorem 5.3 says that strict orthogonality between the partitions F_1, \dots, F_r of Γ gives a design which is optimal among resolved incomplete-block designs. So strict orthogonality is desirable, but it cannot be obtained unless s divides k . What properties of F_1, \dots, F_r give optimal designs when s does not divide k ?

I give a complete answer for $r = 2$ in Section 7. Increasing the replication brings a surprising difficulty in the non-orthogonal case, which I discuss in Section 9. It is easier to characterize optimal designs for the apparently more complicated structure of row-column designs (each superblock is non-trivially a rectangular array), so long as $r = 2$, than for the blocks-in-superblocks structure when $r \geq 3$. I describe what is known about these two situations in Sections 8 and 10 respectively.

6 Balance between two Partitions

As for orthogonal partitions, it is helpful to consider first a general set Δ of size M as in Section 3. For $i = 1, 2$, let F_i be a partition of Δ into m_i sets of size k_i , with corresponding m_i -dimensional subspace V_i and projection matrix P_i . Let W_i be the $(m_i - 1)$ -dimensional subspace of V_i consisting of vectors orthogonal to V_0 (the constant vectors), and put $Q_i = P_i - P_0$ so that Q_i is the matrix of orthogonal projection onto W_i . Then strict orthogonality between the partitions F_1 and F_2 is equivalent to ordinary orthogonality between the spaces W_1 and W_2 . In other words

$$F_1 \perp F_2 \iff Q_1 Q_2 = 0.$$

The second important relation between partitions is defined directly in terms of the matrices Q_i .

Definition Partition F_2 is *balanced* with respect to partition F_1 if there is a scalar θ_{21} such that

$$Q_2 Q_1 Q_2 = \theta_{21} Q_2. \quad (6.1)$$

Thus strict orthogonality is a special case of balance.

(Warning: Pearce [39] and others call this condition *total balance*, presumably to distinguish it from partial balance. I do not think that the word *total* adds anything. James and Wilkinson [28] call it *first order balance*.)

To motivate Equation (6.1), I follow the geometric approach of [28], for which W_1 and W_2 may be any two subspaces of \mathbb{R}^Δ . Consider a vector v in W_2 . The angle α between v and W_1 is defined to be the angle between v and u , where $u = vQ_1 = vQ_2Q_1$. If uQ_2 is a multiple of v , say θv , then this is also the angle between u and W_2 . But in this case v is an eigenvector of $Q_2Q_1Q_2$ with eigenvalue θ , and we see from Figure 6.1 that $\theta = \cos^2 \alpha \in [0, 1]$. (For those who are wary of proof by picture, $uQ_2 = \theta v$, where

$$\theta = \frac{u \cdot v}{v \cdot v} = \frac{u \cdot u + u \cdot (v - u)}{v \cdot v} = \frac{u \cdot u}{v \cdot v}$$

and $0 \leq u \cdot u \leq u \cdot u + (v - u) \cdot (v - u) = v \cdot v$.)

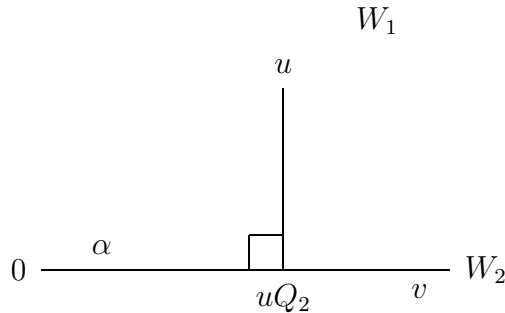


Figure 6.1 Projection of v onto W_1 and u onto W_2

The matrix $Q_2Q_1Q_2$ is symmetric and maps W_2 to itself. Hence W_2 has an orthogonal basis consisting of eigenvectors of $Q_2Q_1Q_2$. The *angles between W_2 and W_1* are defined to be the angles between v and W_1 for vectors v in such a basis. Thus F_2 is balanced with respect to F_1 if and only if all the angles between W_2 and W_1 are the same.

For $i = 1, 2$, let X_i be the $\Delta \times m_i$ matrix whose (δ, Φ) entry is equal to 1 if $F_i(\delta) = \Phi$ and to 0 otherwise. Then $k_i P_i = X_i X_i^T$ and $X_i^T X_i = k_i I_{m_i}$. Put $C_{21} = X_2^T X_1 X_1^T X_2$. If Φ and Ψ are classes of F_2 then the (Φ, Ψ) -entry of C_{21} is equal to $\sum (|\Phi \cap \Xi| \times |\Psi \cap \Xi|)$, the sum being taken over all classes Ξ of F_1 .

Proposition 6.1 *The partition F_2 is balanced with respect to the partition F_1 if and only if $C_{21} \in \mathcal{A}_{m_2}$. Moreover, all eigenvalues of C_{21} lie in $[0, k_1 k_2]$.*

Proof For $i = 1, 2$, we have $Q_i = P_i - P_0 = k_i^{-1} X_i X_i^T - P_0$, and $P_i P_0 = P_0 P_i = P_0$. Thus

$$Q_2 Q_1 Q_2 = k_2^{-2} k_1^{-1} X_2 X_2^T X_1 X_1^T X_2 X_2^T - P_0 = k_2^{-2} k_1^{-1} X_2 C_{21} X_2^T - P_0.$$

If $v \in W_2$ then $v = x X_2^T$ for a unique x in \mathbb{R}^{m_2} , and $v P_0 = 0$, so $x J_{m_2, m_2} = 0$. Thus

$$v Q_2 Q_1 Q_2 = k_2^{-2} k_1^{-1} x X_2^T X_2 C_{21} X_2^T = k_2^{-1} k_1^{-1} x C_{21} X_2^T$$

and v is an eigenvector of $Q_2 Q_1 Q_2$ with eigenvalue θ if and only if x is an eigenvector of C_{21} with eigenvalue $\theta k_1 k_2$. Since the constant vectors are also eigenvectors of C_{21} , the results follow. \diamond

Let us apply these ideas to partitions of Ω , with $F_1 = B$ and $F_2 = f$, so that $k_1 = k$ and $k_2 = r$ and we may denote θ_{21} and C_{21} by θ_{fB} and C_{fB} respectively. If (Ω, B, f) is an incomplete-block design then C_{fB} is precisely the concurrence matrix defined in Section 5, so Proposition 6.1 shows that all eigenvalues of C lie in $[0, rk]$. If (1.1) is not satisfied, the concurrence matrix of a block design is simply defined to be C_{fB} . Then optimality is defined from C as in Section 5, but $\sum \lambda_i$ may no longer be constant. It is immediate from Proposition 6.1 that:

Theorem 6.1 *Let (Ω, B, f) be an incomplete-block design. Then f is balanced with respect to B if and only if (Ω, B, f) is balanced. If it is balanced then*

$$\theta_{fB} = \frac{n - k}{(n - 1)k} = 1 - A. \quad \diamond$$

Even if the incomplete-block design (Ω, B, f) is not balanced, its canonical efficiency factors are $1 - \theta$ for the eigenvalues θ of $Q_f Q_B Q_f$.

However, it is possible to have balance without (1.1) being satisfied.

2	1	1		1		
	2	1	1		1	
		2	1	1		1
1			2	1	1	
	1			2	1	1
1		1			2	1
1	1		1			2

Figure 6.2 A pair of partitions which are balanced but which do not give an optimal block design

1	1		1	1	1	
	1	1		1	1	1
1		1	1		1	1
1	1		1	1		1
1	1	1		1	1	
	1	1	1		1	1
1		1	1	1		1

Figure 6.3 A pair of partitions which are not balanced but which give a better block design

Example 6.1 Consider the array of 35 elements in Figure 6.2. The rows are the classes of F_1 ; the columns are the classes of F_2 . Sizes of non-empty intersections are shown. Then (1.1) is not satisfied because some intersections have size 2.

However, $C_{21} = 4I_7 + 3J_{7,7}$, so F_2 is balanced with respect to F_1 and $\theta_{21} = 4/25$.

Consider the block design on 35 plots whose blocks and treatments are the rows and columns respectively of this array. All of its canonical efficiency factors are equal to $21/25 = 0.84$. Its concurrence matrix is in \mathcal{A}_7 . However, it is not optimal. The incomplete-block design shown in the same format in Figure 6.3 has canonical efficiency factors equal to $(23 - \zeta - \zeta^{-1})/25$ for primitive seventh roots ζ of unity, so $A = 5421/5825 \approx 0.93$ and the design is better than the previous one even though it is not balanced. \diamond

To exclude designs like that in Figure 6.2, I propose the following definition.

Definition Partition F_2 is *strictly balanced* with respect to partition F_1 if F_2 is balanced with respect to F_1 and there is some integer e such that

$$\text{for every class } \Phi \text{ of } F_1 \text{ and every class } \Psi \text{ of } F_2, e \leq |\Phi \cap \Psi| \leq e + 1. \quad (6.2)$$

If F_i has m_i classes of size k_i and $M = m_1 k_1 = m_2 k_2$ then e must equal $\lfloor k_1 k_2 / M \rfloor$. Thus if $k_1 k_2 \leq M$ then condition (6.2) is the translation of condition (1.1) from block designs to pairs of partitions. It is also clearly linked to the characterization of strict orthogonality in Proposition 3.1. (Warning: authors who follow [33] use *balance* for what I am calling *strict balance*.)

To remind the reader that strict orthogonality is a special case of balance, I propose to write $F_i \top F_j$ to indicate that F_i is balanced with respect to F_j , and $F_i \overline{\top} F_j$ to indicate that F_i is strictly balanced with respect to F_j . However, unlike orthogonality, balance is not a symmetric relation: it is possible to have $F_i \top F_j$ but not $F_j \top F_i$. It is clear that $Q_i Q_j Q_i$ cannot be a nonzero multiple of Q_i unless $m_i \leq m_j$. Fisher's Inequality was originally developed for balanced incomplete-block designs in [23], but it applies to any pair of balanced partitions which are not orthogonal. The most helpful version (cf. [14, Theorems 1.14 and 1.15]) is as follows.

Theorem 6.2 (Fisher's Inequality) *Let F_1 and F_2 be partitions of a set Δ into m_1, m_2 classes respectively. Suppose that F_2 is balanced with respect to F_1 but not orthogonal to F_1 . Then $m_2 \leq m_1$, with equality if and only if F_1 is balanced with respect to F_2 , in which case $\theta_{12} = \theta_{21}$.* \diamond

Note that Theorem 6.2 does not need strict balance. It does not even need the classes of either partition to be all of the same size. Some examples are in [40].

Theorem 6.3 *Let (Ω, S, B, f) be a resolved incomplete-block design for sk treatments in r superblocks of s blocks of size k . Then $\dim(W_B Q_f) \leq r(s-1)$.*

Proof Since $S \perp f$, we have $W_S Q_f = 0$. But $W_S \subset W_B$, so

$$\begin{aligned} \dim W_B Q_f &= \dim (W_B \cap W_S^\perp) Q_f \\ &\leq \dim (W_B \cap W_S^\perp) = \dim W_B - \dim W_S = r(s-1). \end{aligned} \quad \diamond$$

Corollary 6.3.1 (Bose's Inequality) *If a resolved incomplete-block design for n treatments in r superblocks of s blocks is balanced then $n-1 \leq r(s-1)$.* \diamond

7 Resolved Incomplete-block Designs with two Superblocks

Let Ω consist of two superblocks of s blocks of size k . We saw in Section 2 that any resolved design for treatment set Γ on the plot set Ω defines partitions F_1 and F_2 of Γ into s classes of size k . From the perspective of Section 6, the triple (Γ, F_1, F_2) is another block design: its plots are the elements of Γ , its blocks are the classes of F_1 and its treatments are the classes of F_2 . It is a square design for s treatments in s blocks of size k . (Strictly speaking, the names of the two superblocks do not matter, so the original resolved design is equivalent to the pair of dual designs (Γ, F_1, F_2) and (Γ, F_2, F_1) .) It is an incomplete-block design if F_1 and F_2 satisfy (6.2) and $k < s$.

Example 7.1 The square design in Figure 1.1 can be written as a square array like those in Figures 6.2 and 6.3. Figure 7.1 shows the 52 plots of Ω as a, b, c, \dots, X, Y, Z . The rows $\alpha, \beta, \dots, \nu$ are the blocks, in the same order as in Figure 1.1. Thus block α is the set of plots $\{a, b, c, d\}$ and so on. The columns are the treatments: for example, treatment 1 is allocated to plots b, e, J and Q .

Now reinterpret this array as a resolved incomplete-block design for treatment set $\{a, b, \dots, Y, Z\}$ in two superblocks of 13 blocks of size 4. In superblock Ω_1 the treatment subsets identified with blocks are $\{a, b, c, d\}, \{e, f, g, h\}$ and so on; while in superblock Ω_2 they are $\{a, F, M, X\}, \{b, e, J, Q\}$ and so on. The resolved design is shown in full in Figure 7.2. \diamond

Williams, Patterson and John [58, 59] call the square block design (Γ, F_1, F_2) the *contraction* of the resolved design. In Example 7.1 the contraction is a square balanced incomplete-block design: equivalently, $F_2 \perp F_1$ (and $F_1 \perp F_2$), since $k < s$.

The utility of the contractions was shown by Patterson and Williams [38], who found the canonical efficiency factors of a resolved design in terms of those of its contraction and hence related the overall efficiency factors of the two designs.

	0	1	2	3	4	5	6	7	8	9	X	E	T
α	a	b			c		d						
β		e	f			g		h					
γ			i	j			k		l				
δ				m	n			o		p			
ε					q	r			s		t		
ζ						u	v			w		x	
η							y	z			A		B
θ	F							C	D			E	
ι		J							G	H			I
κ	M		N							K	L		
λ		Q		R							O	P	
μ			U		V							S	T
ν	X			Y		Z							W

Figure 7.1 The 52 plots in Figure 1.1. Rows are blocks; columns are treatments. See Example 7.1.

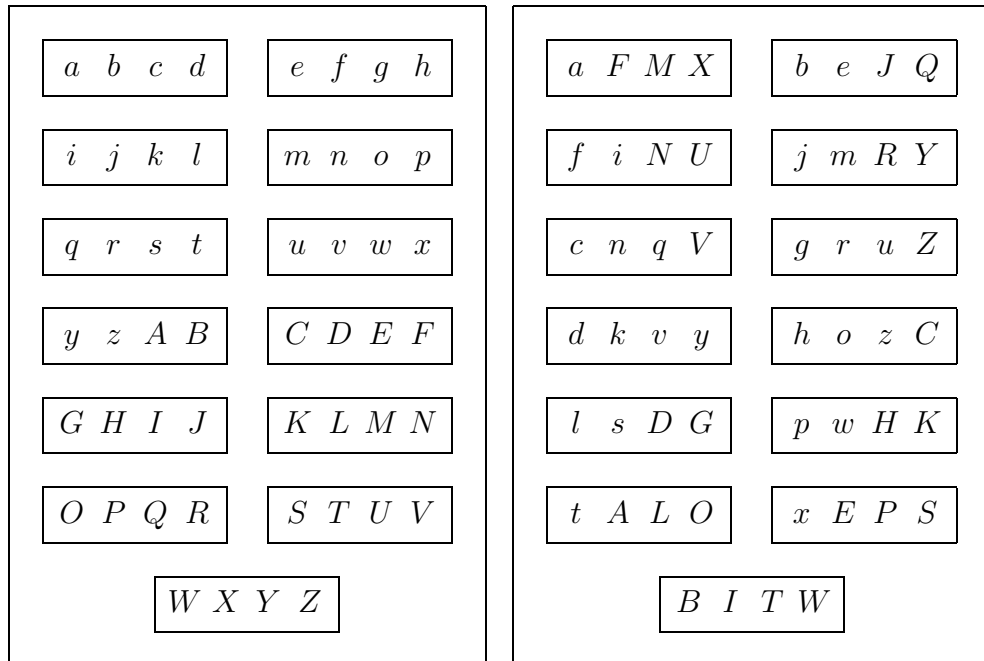


Figure 7.2 Resolved design for 52 treatments in 2 superblocks of 13 blocks of size 4. Its contraction is in Figures 1.1 and 7.1. See Example 7.1.

Theorem 7.1 *Let A be the overall efficiency factor of a resolved incomplete-block design in two superblocks of s blocks of size k , and let A_c be the overall efficiency factor of its contraction. Then*

$$A = \frac{ks - 1}{ks - 2s + 1 + \frac{4(s-1)}{A_c}}. \quad \diamond$$

Corollary 7.1.1 *A resolved incomplete-block design with two superblocks is optimal among resolved incomplete-block designs of that size if and only if its contraction is optimal among square block designs of the relevant size.* \diamond

Theorem 5.1 shows that the square balanced incomplete-block design in Figure 1.1 is optimal. Hence the resolved design in Figure 7.2 is optimal among resolved designs.

As noted in Section 6, if (1.1) is not satisfied then different designs with the same parameters may have different values for $\text{tr } L$. Kiefer [35] extended Theorem 5.1 by proving that any block design for which L is in \mathcal{A}_Γ and has maximal trace is optimal. Therefore any design for which f is strictly balanced with respect to B is optimal. If such a design has $k > n$ then either $n \mid k$ and $B \perp f$ or the design is obtained from a balanced incomplete-block design (possibly disconnected) by adjoining e copies of every treatment to each block, where $e = \lfloor k/n \rfloor$.

Corollary 7.1.2 *If F_1 is strictly balanced with respect to F_2 then the resolved design whose partitions are F_1 and F_2 is optimal among resolved designs.* \diamond

Example 7.2 To obtain an optimal resolved incomplete-block design for 70 treatments in two superblocks of 7 blocks of size 10, start with the square balanced incomplete-block design for 7 treatments in 7 blocks of size 3. Adjoin one copy of each treatment to each block, thus giving 7 blocks of size 10. View both blocks and treatments as partitions of a 70-set into 7 classes of size 10. Use these two partitions to form the blocks in the two superblocks. The resolved design is shown in Figure 7.3:

	1	2	3	4	5	6	7
α	1 2	3 4	5	6 7	8	9	10
β	11	12 13	14 15	16	17 18	19	20
γ	21	22	23 24	25 26	27	28 29	30
δ	31	32	33	34 35	36 37	38	39 40
ε	41 42	43	44	45	46 47	48 49	50
ζ	51	52 53	54	55	56	57 58	59 60
η	61 62	63	64 65	66	67	68	69 70

Figure 7.3 A strictly balanced square block design for 7 treatments in blocks of size 10; also an optimal resolved design for 70 treatments in 2 superblocks of 7 blocks of size 10. See Example 7.2.

the treatments are labelled 1–70; the rows are the blocks of superblock Ω_1 and the columns are the blocks of superblock Ω_2 . \diamond

Bose and Nair [12] constructed resolved incomplete-block designs with two superblocks such that $F_1 \top F_2$. However, they did not insist on strict balance, so some of their designs are not optimal, as was pointed out in [58].

The beauty of Corollary 7.1.1 is that it is not restricted to resolved designs whose contractions are balanced. Every optimal square incomplete-block design translates into a two-replicate resolved incomplete-block design optimal among resolved designs.

When $k = 2$ the only connected square block design for s treatments consists of the edges of an s -cycle. Thus the optimal resolved incomplete-block design has blocks $\{1, 2\}, \{3, 4\}, \dots, \{2s - 1, 2s\}$ in Ω_1 and blocks $\{2, 3\}, \{4, 5\}, \dots, \{2s, 1\}$ in Ω_2 .

If $s = k^2$ and there are $k - 2$ mutually orthogonal $k \times k$ Latin squares then there is a square lattice design for s treatments in s blocks of size k . By Theorem 5.2, this is optimal, even over unresolved designs. Hence it gives an optimal resolved incomplete-block design for k^3 treatments in two superblocks of k^2 blocks of size k . For example, when $k = 4$ such an optimal resolved design may be constructed from its contraction in Figure 4.1(c).

Some other individual square incomplete-block designs are known to be optimal from application of other general theorems.

Example 7.3 Consider the square design for 8 treatments in blocks of size 3 shown in partition form in Figure 7.4. The treatments are 1, \dots , 8; treatments 1, 3 and 8 occur

	1	2	3	4	5	6	7	8
α	a		b					c
β	d			e	f			
γ	g					h	i	
δ		j			k			l
ε		m		n			o	
ζ		p	q			r		
η			s		t		u	
θ				v		w		x

Figure 7.4 A square group-divisible incomplete-block design for 8 treatments in blocks of size 3; also an optimal resolved design for 24 treatments in 2 superblocks of 8 blocks of size 3. See Example 7.3.

in block α and so on. Divide the treatments into the four groups $\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}$. Then the concurrence of any two treatments in the same group is 0, while that of any two treatments in different groups is 1 (so this is a special sort of *group-divisible design*: see [53, Sections 8.4–8.6]). The Corollary to Theorem 3.1 of [15] shows that this is optimal. Hence an optimal resolved design for the 24 treatments a, b, \dots, x in 16 blocks of size 3 is obtained by using the rows and columns of Figure 7.4 as blocks in the two superblocks. \diamond

Example 7.4 Theorem 2.2 of [16] shows that an optimal square design for 15 treatments in blocks of size 3 is the *triangular* design (see [53, Sections 8.7–8.8]) whose treatments are the 2-subsets of $\{1, \dots, 6\}$ and whose blocks are the *synthemes* such as $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ (see [14, page 81]). This gives an optimal resolved design for 45 treatments in 30 blocks of size 3. \diamond

To summarize, Corollary 7.1.1 gives a complete answer to the question: When $r = 2$, what properties of F_1, F_2 make a resolved design optimal? The answer is: Optimality of the square incomplete-block design (Γ, F_1, F_2) , a special case of which is strict balance between F_1 and F_2 .

8 Resolved Row-column Designs with two Superblocks

In a nested row-column design, Ω consists of r superblocks each of which is a $k \times s$ rectangular array. Let R and B be the partitions of Ω into rk rows and rs columns respectively. Then R, B and S (the superblock partition) are pairwise orthogonal but not strictly orthogonal. As always, the design also includes a map f from Ω to the treatment set Γ , and the design is resolved if the restriction f_i of f to Ω_i is a bijection for $i = 1, \dots, r$.

As before, let F_i be the partition of Γ induced by f_i and B , and let G_i be the partition of Γ induced by f_i and R . Then $F_i \perp G_i$ for $i = 1, \dots, r$. If $s = k$ it is possible to have pairwise strict orthogonality among all of $F_1, \dots, F_r, G_1, \dots, G_r$. Then the design is a *lattice square* design, introduced by Yates in [62]. Such a design may be constructed from a square lattice design with $2r$ superblocks: pair the old superblocks in any manner and use each pair of partitions to form one superblock of the new design.

Example 8.1 Figure 8.1 shows a lattice square design for 16 treatments in two superblocks. The first superblock is obtained from the first two superblocks of Figure 4.1(c), the second from the last two. \diamond

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

a	f	k	p
g	d	m	j
l	o	b	e
n	i	h	c

Figure 8.1 Lattice square design constructed from the design in Figure 4.1(c). See Example 8.1.

Pearce [39] extended a matrix notation introduced in [26] for showing pairwise relationships among a set of partitions. The partitions label the rows and columns

of the matrix. The (F, G) -entry is – if $F = G$; it is O if $F \perp G$; it is T if $G \overline{\top} F$ but $G \not\perp F$ and T' if $F \overline{\top} G$ but $G \not\perp F$. In this notation the relationships among the partitions in a lattice square design with two superblocks are

$$\begin{array}{c} F_1 \\ G_1 \\ F_2 \\ G_2 \end{array} \begin{bmatrix} F_1 & G_1 & F_2 & G_2 \\ - & O & O & O \\ O & - & O & O \\ O & O & - & O \\ O & O & O & - \end{bmatrix}$$

It is possible to make lattice square designs for larger values of r so long as there are $2r - 2$ mutually orthogonal $k \times k$ Latin squares.

When $s \neq k$ it is not possible to have $F_i \perp F_j$ for $i \neq j$. It *may* be possible to have $F_i \perp G_j$ for $i \neq j$. We now restrict attention to $r = 2$, as in the previous section.

The nicest pairwise relationships on $\{F_1, G_1, F_2, G_2\}$ would be

$$\begin{array}{c} F_1 \\ G_1 \\ F_2 \\ G_2 \end{array} \begin{bmatrix} F_1 & G_1 & F_2 & G_2 \\ - & O & T & O \\ O & - & O & T \\ T & O & - & O \\ O & T & O & - \end{bmatrix}$$

(Note that, because F_1 and F_2 have the same number of classes, $F_1 \overline{\top} F_2$ if and only if $F_2 \overline{\top} F_1$; and similarly for G_1 and G_2 .)

Definition A *double Youden rectangle* of size $k \times s$ is a set Γ of order ks with partitions F_1 and F_2 into s sets of size k and partitions G_1 and G_2 into k sets of size s such that $F_i \perp G_j$ for $i, j \in \{1, 2\}$ and $F_1 \overline{\top} F_2$ and $G_1 \overline{\top} G_2$.

Example 8.2 A double Youden rectangle of size 4×5 is shown in Figure 8.2 in two formats. Figure 8.2(a) shows the nested row-column design.

The treatment set Γ is $\{a, b, \dots, t\}$; the partitions G_1 and F_1 are the rows and columns of the first superblock Ω_1 ; the partitions G_2 and F_2 are the rows and columns of the second superblock Ω_2 .

To see that F_1 is strictly orthogonal to G_2 , choose any column of Ω_1 , say $\{a, b, c, d\}$, and observe where these four treatments occur in Ω_2 . They are in different rows of Ω_2 . Moreover, they occur in four different columns of Ω_2 , leaving just one column unaccounted for. A similar pattern arises for every column of Ω_1 , and this shows that $F_1 \perp G_2$ and $F_1 \overline{\top} F_2$.

Now consider a row of Ω_1 , say $\{a, i, e, m, q\}$. These five treatments occur in different columns of Ω_2 . Two of them occur in the first row of Ω_2 , the remainder in different rows. As a similar pattern arises for every row of Ω_1 , this shows that $G_1 \perp F_2$ and $G_1 \overline{\top} G_2$.

Figure 8.2(a) is a picture of Ω , showing how treatments are allocated to the 40 plots, which *already* have the partitions S , B and R .

<i>a</i>	<i>e</i>	<i>i</i>	<i>m</i>	<i>q</i>
<i>b</i>	<i>f</i>	<i>j</i>	<i>n</i>	<i>r</i>
<i>c</i>	<i>g</i>	<i>k</i>	<i>o</i>	<i>s</i>
<i>d</i>	<i>h</i>	<i>l</i>	<i>p</i>	<i>t</i>

<i>e</i>	<i>a</i>	<i>s</i>	<i>l</i>	<i>n</i>
<i>j</i>	<i>p</i>	<i>b</i>	<i>q</i>	<i>g</i>
<i>o</i>	<i>r</i>	<i>h</i>	<i>c</i>	<i>i</i>
<i>t</i>	<i>k</i>	<i>m</i>	<i>f</i>	<i>d</i>

(a) A resolved nested row-column design for 20 treatments

1 <i>B</i> <i>a</i>	1 <i>A</i> <i>e</i>	3 <i>E</i> <i>i</i>	4 <i>C</i> <i>m</i>	2 <i>D</i> <i>q</i>
2 <i>C</i> <i>b</i>	4 <i>D</i> <i>f</i>	2 <i>A</i> <i>j</i>	1 <i>E</i> <i>n</i>	3 <i>B</i> <i>r</i>
3 <i>D</i> <i>c</i>	2 <i>E</i> <i>g</i>	4 <i>B</i> <i>k</i>	3 <i>A</i> <i>o</i>	1 <i>C</i> <i>s</i>
4 <i>E</i> <i>d</i>	3 <i>C</i> <i>h</i>	1 <i>D</i> <i>l</i>	2 <i>B</i> <i>p</i>	4 <i>A</i> <i>t</i>

(b) Four partitions of Γ

Figure 8.2 Two views of the double Youden rectangle in Example 8.2

An alternative view is in Figure 8.2(b). This is a picture of Γ . The rows and columns are the partitions G_1 and F_1 from Ω_1 : for example, treatment h is in row 4, column 2. The numbers 1, ..., 4 are the classes of G_2 , that is, the rows of Ω_2 . For example, $G_2(h) = 3$. Similarly, the letters A, \dots, E are the classes of F_2 , that is, the columns of Ω_2 .

The relationship between Figures 8.2(a) and (b) is comparable to that between Figures 7.2 and 7.1. \diamond

The labelling of the elements of Γ in Figure 8.2(b) is helpful to show the equivalence of Figures 8.2(a) and (b), but unnecessary to show the family of four partitions on Γ . A double Youden rectangle is therefore usually displayed as a $k \times s$ rectangle with two symbols in each cell, one from a k -set and the other from an s -set.

The design in Figure 8.2 is constructed by a method of [25], which gives double Youden rectangles of size $k \times (k + 1)$ for all $k \notin \{2, 3, 6\}$. The method uses a pair of mutually orthogonal $k \times k$ Latin squares with a common transversal. The 6×7 double Youden rectangle in Figure 8.3 is given in [43], where it is attributed to G. H. Freeman. The 4×13 double Youden rectangle in Figure 8.4 was found by Preece [45]: it is attractive because it can be displayed using an ordinary pack of cards. Further double Youden rectangles are given in [43, 47, 48, 49, 51, 52].

<i>C</i> α	<i>G</i> β	<i>E</i> γ	<i>F</i> δ	<i>A</i> ϵ	<i>D</i> ζ	<i>B</i> η
<i>E</i> β	<i>D</i> γ	<i>G</i> ζ	<i>A</i> α	<i>C</i> δ	<i>B</i> ϵ	<i>F</i> β
<i>F</i> γ	<i>C</i> ζ	<i>A</i> β	<i>G</i> ϵ	<i>B</i> α	<i>E</i> δ	<i>D</i> δ
<i>G</i> δ	<i>E</i> ϵ	<i>D</i> α	<i>B</i> β	<i>F</i> ζ	<i>A</i> γ	<i>C</i> ϵ
<i>D</i> ϵ	<i>F</i> α	<i>B</i> δ	<i>E</i> ζ	<i>G</i> γ	<i>C</i> β	<i>A</i> ζ
<i>B</i> ζ	<i>A</i> δ	<i>F</i> ϵ	<i>C</i> γ	<i>D</i> β	<i>G</i> α	<i>E</i> α

Figure 8.3 A 6×7 double Youden rectangle

However many superblocks it has, a nested row-column design is defined in [21, 22] to have *adjusted orthogonality* if the projections of W_R and W_B onto W_f are orthog-

A ♠	3 ♣	4 ♥	7 ♥	8 ♣	2 ♣	10 ♦	J ♠	5 ♠	6 ♦	Q ♦	K ♠	9 ♥
2 ♦	5 ♥	3 ♦	4 ♠	6 ♠	7 ♦	8 ♠	9 ♣	10 ♣	K ♥	J ♥	Q ♣	A ♦
4 ♣	J ♦	6 ♣	K ♦	5 ♦	9 ♠	7 ♣	8 ♥	Q ♥	10 ♠	A ♣	2 ♥	3 ♠
10 ♥	2 ♠	Q ♠	5 ♣	A ♥	6 ♥	3 ♥	4 ♦	9 ♦	J ♣	7 ♠	8 ♦	K ♣

Figure 8.4 A 4×13 double Youden rectangle

onal subspaces of \mathbb{R}^Ω . (Regrettably, some authors call such a design ‘orthogonal’.) Equivalent characterizations are

$$F_i \perp G_j \quad \text{whenever } i \neq j \quad (8.1)$$

and

$$\text{if } x \text{ is an eigenvector of } C_{fB} \text{ with non-zero eigenvalue then } xC_{fR} = 0: \quad (8.2)$$

see [3, 37]. Here, as in Section 6, C_{fB} denotes the concurrence matrix for treatments in the partition B , that is, in the incomplete-block design whose blocks are columns, which is sometimes called the *column component design*. Similarly, C_{fR} is the concurrence matrix for the row component design.

Are double Youden rectangles optimal among resolved nested row-column designs? The matrix L whose eigenvalues are the canonical efficiency factors of the design is given by

$$\begin{aligned} L &= I_\Gamma - (rk)^{-1}C_{fB} - (rs)^{-1}C_{fR} + n^{-1}J_{\Gamma,\Gamma} \\ &= L_B + L_R - I_\Gamma + n^{-1}J_{\Gamma,\Gamma}, \end{aligned}$$

where L_B and L_R are the matrices for the column component design and the row component design respectively. Calculation is simplified if L_B commutes with L_R , a condition which is called *general balance*: see [3, 37]. Condition (8.2) is a special case of general balance.

Theorem 8.1 ([20, 31]) *Let A be the overall efficiency factor of a resolved nested row-column design with general balance, and let A_R and A_B be the overall efficiency factors of its row component design and column component design respectively. Then*

$$A \leq \frac{1}{\frac{1}{A_R} + \frac{1}{A_B} - 1}$$

with equality if and only if the design has adjusted orthogonality. \diamond

Corollary 8.1.1 *If the row component design and the column component design of a resolved nested row-column design with adjusted orthogonality are both optimal among resolved incomplete-block designs of the appropriate sizes then the nested row-column design is optimal among resolved nested row-column designs with general balance.* \diamond

Corollary 8.1.2 *Lattice square designs are optimal amongst resolved nested row-column designs with general balance.* \diamond

Corollary 8.1.3 *Double Youden rectangles are optimal amongst generally balanced resolved nested row-column designs with two superblocks.* \diamond

The converse of Corollary 8.1.1 is not true, as it is not always possible to fit together optimal component designs into a resolved nested row-column design (for example, consider $r = 2$ and $s = k = 6$). However, the beauty of Corollary 8.1.1 is that we can apply it to any optimal component designs, not just balanced ones. When $r = 2$ we use Corollary 7.1.1 to verify that the component designs are optimal.

The conditions in Corollary 8.1.1 cannot be satisfied if $k = 2$, because strict orthogonality between F_j and both G_1 and G_2 forces the column component design to be disconnected. They also seem unlikely to be satisfied unless

$$r(s + k - 2) \leq sk - 1. \quad (8.3)$$

Adjusted orthogonality means that $W_R Q_f \perp W_B Q_f$ and hence

$$\dim(W_R Q_f) + \dim(W_B Q_f) \leq n - 1.$$

However, an efficient column component design will usually have a high value of $\dim(W_B Q_f)$: if it is balanced then $\dim(W_B Q_f) = n - 1$. If $r(s - 1) < n - 1$ then Theorem 6.3 shows that at least $n - 1 - r(s - 1)$ of the angles between W_f and W_B must be right angles (corresponding to eigenvalues $\theta = 0$). Since we want to maximize the harmonic mean of the $1 - \theta$, we do not want any more than necessary of the θ to be zero, so it seems likely that $\dim(W_B Q_f) = r(s - 1)$ if (Ω, S, B, f) is optimal. This heuristic reasoning is not quite exact, but $\dim(W_B Q_f)$ does not often fall much below $r(s - 1)$ in optimal resolved designs.

Example 8.3 Figure 4.2 shows a set of four pairwise strictly orthogonal partitions on a set Γ of 16 treatments. A fifth such partition F_5 can be added: it corresponds to a Latin square orthogonal to those in Figure 4.1(b). In Example 8.1, Ω_1 is (Γ, F_1, F_2) and Ω_2 is (Γ, F_3, F_4) . Adjoining superblocks (Γ, F_5, F_1) , (Γ, F_2, F_3) and (Γ, F_4, F_5) gives a design which is optimal because both of its component designs are balanced. However, it does not have adjusted orthogonality. Moreover, no design with adjusted orthogonality is as good. But $r(s + k - 2)$ is twice as big as $sk - 1$ here. \diamond

Preece [42] gives some resolved nested row-column designs both of whose component designs are balanced. There are also resolved designs where L is in \mathcal{A}_Γ even though L_R and L_B are not: for example, replace the fifth superblock in Example 8.3 by (Γ, F_5, F_4) . Some examples are in [55]. They tend to have rather large values of r .

All the same, it seems plausible that if Equation (8.3) is satisfied then the requirement for general balance could be removed from the competing designs in Corollaries 8.1.1–8.1.3. That would seem to combine the results of Sections 5 and 7. However,

I have not seen a proof of this. Moreover, I shall show in the next section that optimality properties of three or more partitions do not always follow from their pairwise properties, so the result may be false.

However, we can draw a parallel with Theorem 7.1. As shown in Figure 8.2, the four partitions of Γ in a resolved nested row-column design with two superblocks can be regarded as a row-column design for two sets of treatments: there are k rows and s columns, one set of treatments has size k and the other has size s . Bailey and Patterson [7] call this the *contraction* of the nested row-column design. Its existence does not depend on adjusted orthogonality. Efficiency factors and optimality can also be defined for a design for two sets of treatments. Jarrett, Piper and Wild [29] calculated the canonical efficiency factors of the resolved nested row-column design in terms of those of its contraction and proved:

Theorem 8.2 *A resolved nested row-column design with two superblocks is optimal among such designs if and only if its contraction is optimal.* \diamond

John and Williams [32] compare three algorithms for constructing resolved nested row-column designs with two superblocks. They find that the algorithm which uses contractions performs consistently best both for speed of execution and for overall efficiency factor of the design constructed.

9 Balance among three or more Partitions

What is the appropriate generalization of balance to three or more partitions? For statistical purposes, what is important for the partition F_i is the angle or angles between W_i and the vector space sum of the other W_j .

Definition Suppose that F_j is a partition of Δ for $j \in K \cup \{i\}$, where $i \notin K$. Then F_i is *balanced with respect to* $\{F_j : j \in K\}$ if the angles between W_i and W_K are all the same, where $W_K = \sum_{j \in K} W_j$; that is, if there is a scalar ϕ_{iK} such that $Q_i Q_K Q_i = \phi_{iK} Q_i$, where Q_K is the matrix of orthogonal projection onto W_K . In particular, F_i is balanced with respect to F_j and F_l if $Q_i Q_{jl} Q_i = \phi_{i,jl} Q_i$, where Q_{jl} is the matrix of orthogonal projection onto $W_j + W_l$.

Preece [44] calls this *overall total balance*: I think that it is important to emphasize what is balanced with respect to what.

Suppose that $F_j \perp F_l$. Then $Q_{jl} = Q_j + Q_l$, and so $Q_i Q_{jl} Q_i = Q_i Q_j Q_i + Q_i Q_l Q_i$: thus F_i is balanced with respect to F_j and F_l if it is balanced with respect to each of F_j and F_l separately. Now suppose that $F_i \perp F_j$. Then $Q_i Q_{jl} Q_i = Q_i Q_l Q_i$, and so F_i is balanced with respect to F_j and F_l if and only if it is balanced with respect to F_l . Therefore, so long as

$$\text{there is at least one strictly orthogonal pair in each trio of partitions,} \quad (9.1)$$

pairwise balance implies overall balance.

If each F_j has the same number of classes, all of the same size, then it is tempting to think that the pairwise balance summarized by

$$\begin{matrix} & F_1 & F_2 & F_3 \\ F_1 & \begin{bmatrix} - & T & T \end{bmatrix} \\ F_2 & \begin{bmatrix} T & - & T \end{bmatrix} \\ F_3 & \begin{bmatrix} T & T & - \end{bmatrix} \end{matrix}$$

is sufficient for overall balance. It is not.

Example 9.1 The set of three partitions in Figure 9.1 is adapted from an example in [26]. Each partition has 7 sets of size 4. The rows and columns are the classes of two of the partitions. The letters A, \dots, G are the classes of the third partition; their presence also indicates the 28 elements of the underlying set. Each pair of partitions is strictly balanced, but none of the three partitions is balanced with respect to the other two, because Equation (9.3) below is not satisfied. \diamond

G	A	F		D		
A	G		B			E
F		A			B	C
	B			C	D	G
D			C	A	E	
		B	D	E		F
	E	C	G		F	

Figure 9.1 Three partitions with pairwise strict balance but without overall balance. See Example 9.1.

See [46] for more examples. Pearce's matrix notation simply does not include enough information unless (9.1) is satisfied, although Preece made a valiant attempt to extend it in [44].

To investigate further, we must evaluate Q_{jl} .

Theorem 9.1 (James [27]) *Let W_1 and W_2 be subspaces of \mathbb{R}^Δ such that*

$$Q_1 Q_2 Q_1 = \theta Q_1,$$

where Q_i is the matrix of orthogonal projection onto W_i for $i = 1, 2$. If $\theta \neq 0$ the matrix of orthogonal projection onto $W_1 + W_2$ is

$$Q_2 + \frac{Q_1 - Q_2 Q_1 - Q_1 Q_2 + Q_2 Q_1 Q_2}{1 - \theta}.$$

If $\dim W_1 = \dim W_2$ this simplifies to

$$\frac{Q_1 + Q_2 - Q_2 Q_1 - Q_1 Q_2}{1 - \theta}.$$

\diamond

Corollary 9.1.1 *Suppose that each of F_i, F_j, F_l is balanced with respect to each of the others. Then F_i is balanced with respect to F_j and F_l if and only if*

$$X_i^T X_j X_j^T X_l X_l^T X_i + X_i^T X_l X_l^T X_j X_j^T X_i \in \mathcal{A}_m, \quad (9.2)$$

where m is the number of classes of each of F_i, F_j, F_l . \diamond

Write $N_{jl} = X_j^T X_l$. This is called the *incidence matrix* of F_j in F_l . If Φ and Ψ are classes of F_j and F_l respectively then the (Φ, Ψ) -entry of N_{jl} is equal to $|\Phi \cap \Psi|$. Note that $N_{lj} = N_{jl}^T$ and $N_{jl} N_{lj} = C_{jl}$. Now condition (9.2) can be rewritten as

$$N_{ij} N_{jl} N_{li} + N_{il} N_{lj} N_{ji} \in \mathcal{A}_m. \quad (9.3)$$

An appealing combinatorial property suggested by (9.3) is

$$N_{ij} N_{jl} = c_{il} N_{il} + d_{il} J_m \quad \text{for some scalars } c_{il} \text{ and } d_{il}. \quad (\mathcal{P}(i, j, l))$$

For each class Φ of F_i and each class Ψ of F_l the (Φ, Ψ) -entry of $N_{ij} N_{jl}$ is equal to $\sum |\Phi \cap \Xi| \times |\Psi \cap \Xi|$, the sum being taken over all classes Ξ of F_j . If $k < m$ and each pair of F_i, F_j, F_l satisfies (1.1) then $\mathcal{P}(i, j, l)$ implies that the number of Ξ for which $\Phi \cap \Xi$ and $\Psi \cap \Xi$ are both non-empty is equal to d_{il} if $\Phi \cap \Psi$ is empty and to $c_{il} + d_{il}$ otherwise. Calculation of row sums of the matrices shows that $c_{il}k + d_{il}m = k^2$.

Theorem 9.2 *Let F_1, F_2, F_3 be partitions of Δ into m sets of size k , where $k < m$. Assume that each pair of F_1, F_2, F_3 is strictly balanced. If $\mathcal{P}(i, j, l)$ is satisfied for any ordering (i, j, l) of $(1, 2, 3)$ then it is satisfied for all of them and $c_{gh} = c$, $d_{gh} = d$ for every 2-subset $\{g, h\}$ of $\{1, 2, 3\}$, where $c^2 = k(m - k)/(m - 1)$ and $d = k(k - c)/m$.*

Proof Note first that $\mathcal{P}(i, j, l)$ implies $\mathcal{P}(l, j, i)$ with $c_{li} = c_{il}$, $d_{li} = d_{il}$.

Assume that $N_{12} N_{23} = c_{13} N_{13} + d_{13} J_m$. Then $N_{12} N_{23} N_{32} = c_{13} N_{13} N_{32} + d_{13} k J_m$. But $N_{23} N_{32} = C_{23} = [k(m - k) I_m + k(k - 1) J_m] / (m - 1)$, so

$$N_{13} N_{32} = c_{12} N_{12} + d_{12} J_m$$

with

$$c_{12} = \frac{k(m - k)}{(m - 1)c_{13}} \quad \text{and} \quad d_{12} = \frac{k}{c_{13}} \left[\frac{k(k - 1)}{m - 1} - d_{13} \right].$$

Thus $\mathcal{P}(i, j, l)$ is preserved by transposition of the outer coordinates and by transposition of the last two coordinates, so it is preserved by all permutations of the three coordinates. Furthermore,

$$c_{32} = c_{23} = \frac{k(m - k)}{(m - 1)c_{21}} = \frac{k(m - k)}{(m - 1)c_{12}} = c_{13} = c_{31}$$

so all the c_{gh} are equal. \diamond

(In fact, $k < m$ is not necessary for the main result of Theorem 9.2. Strict balance gives $c^2 = p$ where $C = pI + qJ$.)

Lemma 9.1 *Condition $\mathcal{P}(i, j, l)$ implies that $Q_i Q_j Q_l$ is a multiple of $Q_i Q_l$.*

Proof We have

$$\begin{aligned}
Q_i Q_j Q_l &= k^{-3} X_i X_i^T X_j X_j^T X_l X_l^T - P_0 \\
&= k^{-3} X_i N_{ij} N_{jl} X_l^T - P_0 \\
&= k^{-3} X_i [cN_{il} + dJ_m] X_l^T - P_0 \\
&= k^{-3} c X_i X_i^T X_l X_l^T + k^{-3} d J_m - P_0 \\
&= k^{-1} c P_i P_l + k^{-2} d m P_0 - P_0 \\
&= k^{-1} c Q_i Q_l + k^{-1} c P_0 + k^{-2} d m P_0 - P_0 \\
&= k^{-1} c Q_i Q_l.
\end{aligned}$$

◇

Theorem 9.3 *Let $\{F_j : j \in K\}$ be a set of partitions of Δ into m classes of size k , such that each pair is balanced and $\mathcal{P}(i, j, l)$ is satisfied for each trio. Then*

- (i) F_i is balanced with respect to $\{F_j : j \in K, j \neq i\}$ for all i in K ;
- (ii) the matrix of orthogonal projection onto $\sum_{j \in K} W_j$ is a linear combination of the Q_j and the products $Q_j Q_l$.

Proof We use induction on $|K|$. If $|K| = 2$ then the pairwise balance is (i) and Theorem 9.1 gives (ii). Now assume (ii) for $K \setminus \{i\}$. Then $Q_i Q_{K \setminus \{i\}} Q_i$ is a linear combination of the $Q_i Q_j Q_i$ and the $Q_i Q_j Q_l Q_i$ for $j, l \in K \setminus \{i\}$. Pairwise balance shows that $Q_i Q_j Q_i$ is a multiple of Q_i ; and two applications of Lemma 9.1 show that $Q_i Q_j Q_l Q_i$ is a multiple of Q_i . This proves (i) for K .

Now apply Theorem 9.1 to $W_{K \setminus \{i\}}$ and W_i . By (ii) for $K \setminus \{i\}$, the matrix Q_K is a linear combination of products of at most three of the Q_j . Lemma 9.1 now gives (ii) for K . ◇

The proof of Theorem 9.3 shows that sets of partitions with pairwise balance and every $\mathcal{P}(i, j, l)$ satisfied also have *setwise balance*, which means that each F_i is balanced with respect to $\{F_j : j \in \tilde{K}\}$ for every subset \tilde{K} of $K \setminus \{i\}$.

Consider a set of $r - 2$ mutually orthogonal $t \times t$ Latin squares with a common transversal. Let Δ be the $t \times t$ square array with the cells on the transversal removed. Let F_1 and F_2 be the partitions of Δ into rows and columns. For $i \geq 3$, let F_i be the partition of Δ given by the letters of the $(i - 2)$ -th Latin square. In each partition, label each class by the name of the transversal cell which it would contain if the transversal were restored. Then $N_{ij} = J - I$ for all $i \neq j$ so each partition is balanced with respect to every subset of the others. Here $k = t - 1$, $m = t$, $c = -1$ and $d = t - 1$.

Example 9.2 Consider the array in Figure 4.1(a) and the second Latin square in Figure 4.1(b). The main diagonal is a transversal. Removing it gives the three partitions in Figure 9.2. \diamond

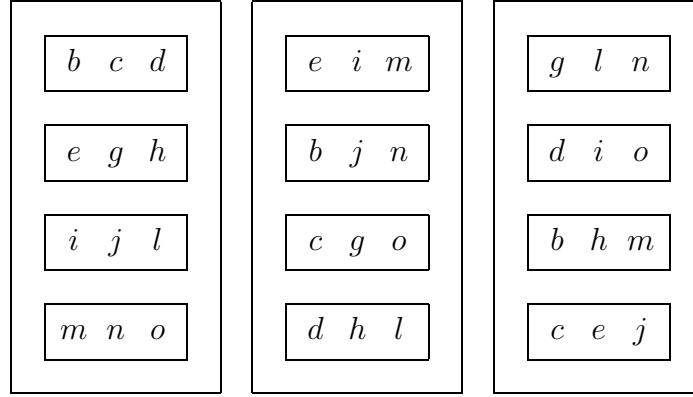


Figure 9.2 Three partitions which satisfy $\mathcal{P}(1, 2, 3)$; also a rectangular lattice design for 12 treatments in 3 superblocks of 4 blocks of size 3. See Example 9.2.

Similarly, by doubling the transversal rather than removing it, we obtain a set of partitions with $m = t$, $k = t + 1$ and $N_{ij} = I + J$. Once again the partitions have setwise balance. Here $c = 1$ and $d = t + 1$. An example with $t = r = 3$ is in Figure 1.2.

Preece and Cameron [50] give a set of four partitions satisfying the hypotheses of Theorem 9.3 with $m = 16$, $k = 6$, $c = -2$ and $d = 3$.

For trios of partitions which satisfy (9.2) but not $\mathcal{P}(i, j, l)$, an analogue of Theorem 9.2 holds.

Theorem 9.4 Let F_1, F_2, F_3 be partitions of Δ into m sets of size k . Suppose that each pair of F_1, F_2, F_3 is strictly balanced but not orthogonal. If there are scalars y, z such that

$$N_{12}N_{23}N_{31} + N_{13}N_{32}N_{21} = yI + zJ \quad (9.5)$$

then $N_{23}N_{31}N_{12} + N_{21}N_{13}N_{32} = N_{31}N_{12}N_{23} + N_{32}N_{21}N_{13} = yI + zJ$.

Proof Examination of the row sums of the matrices in Equation (9.5) shows that $y + mz = 2k^3$. Pre-multiplication of Equation (9.5) by N_{21} and post-multiplication by N_{12} gives

$$C_{21}N_{23}N_{31}N_{12} + N_{21}N_{13}N_{32}C_{21} = yC_{21} + zk^2J.$$

Strict balance without orthogonality means that there are nonzero scalars p and q , with $p + mq = k^2$, such that $C_{21} = pI + qJ$, so

$$p(N_{23}N_{31}N_{12} + N_{21}N_{13}N_{32}) + 2qk^3J = pyI + yqJ + zk^2J,$$

whence $p(N_{23}N_{31}N_{12} + N_{21}N_{13}N_{32}) = pyI + pzJ$. \diamond

Bailey, Preece and Rowley [8] give an infinite series of such sets of partitions. The values of m are all prime powers congruent to 3 modulo 4; and $k = (m \pm 1)/2$. In each set, every pair of partitions is strictly balanced and satisfies (1.1). So it is surprising to find that there are two different values among the $\phi_{i,jl}$. Unlike θ in Theorem 6.1, the values of ϕ are not determined by the values of m and k together with properties of incompleteness and balance. This surprising possibility had already been noted in individual cases in [19, 41, 1].

Although the sets of partitions in [8] do not satisfy $\mathcal{P}(i, j, l)$, they do have setwise balance; in particular they satisfy Theorem 9.3(ii). Just as for the partitions with $m = k \pm 1$, the proof of this depends on a natural labelling of the classes of every partition by the same m -set, so that it makes sense to talk of different matrices N_{ij} being “the same” even though their rows and columns are labelled by classes of different partitions. There is a particular matrix $N \in \mathbb{R}^{m \times m}$ such that (a) every N_{ij} is equal to N or N^T and (b) both N^2 and N^T are linear combinations of I , N and J . These conditions ensure that not only is (9.3) satisfied: so are its higher-order analogues. Setwise balance can then be proved inductively.

10 Resolved Incomplete-block Designs with three or more Superblocks

Corollary 7.1.2 suggests that it may be fruitful to examine resolved designs for three or more superblocks whose partitions F_1, \dots, F_r have some sort of balance among them. The most promising property seems to be that each F_i be balanced with respect to all of the others together. All the sets of partitions in Section 9 which satisfy this also satisfy the stronger condition of setwise balance, that is, that each F_i is balanced with respect to every subset of the other F_j .

This stronger condition could be an advantage, because it remains true if any superblock is lost from the design. Such loss is not unknown in real experiments. Superblocks often correspond to time or space. If conditions prevent an experiment being continued to its planned conclusion, it will usually be one or more whole superblocks that are omitted. If a spatial experiment has been laid out well, limited damage from pests or other outside influences should be confined to one or two superblocks.

Consider the set of r partitions into $k + 1$ sets of size k described immediately after Theorem 9.3. This gives a resolved design for $k(k + 1)$ treatments in $r(k + 1)$ blocks of size k . Such designs are called *rectangular lattices*. They were introduced by Harshbarger in [24] and are quite widely used. They have high overall efficiency factors (see [57, 18, 9]) but have not been proven optimal except for $r = 2$.

The simple generalization of rectangular lattices exemplified in Figure 1.2 can be carried further. Hence we obtain resolved designs for sk treatments whenever $k \pm 1$ is a multiple of s .

The partitions in [50] give resolved designs for 96 treatments in blocks of size 6 in up to four superblocks.

Resolved designs constructed from the partitions in [8] are discussed in [4]. Only three superblocks are considered. Even this small value of r presents a difficulty

which can only get worse as r increases. The value of the overall efficiency factor A depends on the value of ϕ . Since there are two possible values of ϕ , the balance properties alone cannot guarantee optimality.

When $s = 2$, each F_i is trivially balanced with respect to every subset of the rest, so something more is needed for optimality.

So just what does make a resolved design efficient if $r \geq 3$, $s \geq 3$ and s does not divide k ? Setwise balance is very appealing, but cannot be sufficient when there is more than one possible value of ϕ . Would it be better to have each F_i balanced with respect to all of the others together but not balanced with respect to all subsets of the others?

Are there any such sets of partitions? Yes: any balanced resolved incomplete-block design which is not affine resolved, such as resolvable unreduced designs, which are discussed in [13]. (There is an interesting resolved unreduced design for 8 treatments in 28 blocks of size 2: each pair of partitions is non-strictly orthogonal but there are two relationships among trios.) But these have high replication (from Bose's Inequality), and we already know their optimality from Theorem 5.1. Are there any such sets with more realistic values of r , such as 3, 4, 5? A search for them will have to jettison pairwise balance, so many existing combinatorial techniques will not be helpful.

I feel that there should be some purely combinatorial condition on sets of three partitions which gives an analogue of Corollary 7.1.1, but I do not know what it is.

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