Rectangular experiments: restricted randomization or row-column designs?

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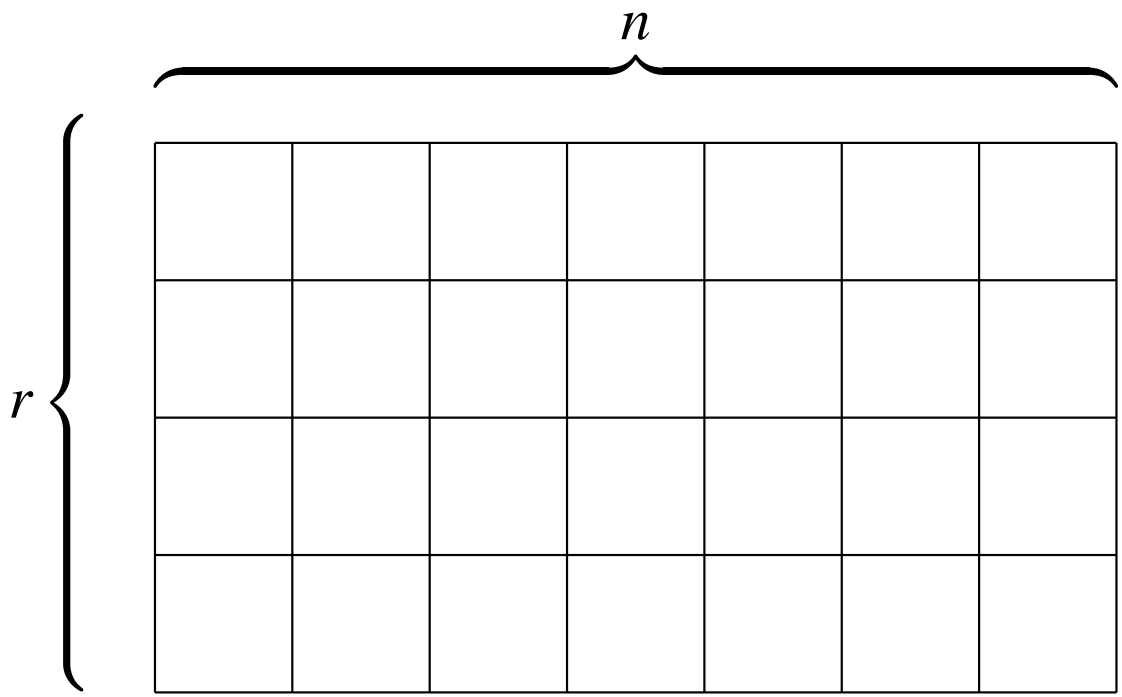
Thanks to CAPES for support in Brasil

The problem

An agricultural experiment to compare n treatments.
The experimental area has r rows and n columns.



Use a randomized complete-block design with rows as blocks.
(In each row, choose one of the $n!$ orders with equal probability.)

What should we do if the randomization produces a plan with one treatment always at one side of the rectangle?

Example

Federer (1955 book): guayule trees

B	D	G	A	F	C	E
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G	E	D	F	B	C	A
B	A	C	F	G	E	D
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Assumed model

Y_α is the response on plot α.

$E(Y_\alpha) = \theta_i$ where i is the treatment on α.

$$\begin{aligned} \mathrm{Var}(Y_\alpha) &= \sigma^2 \qquad \text{for all } \alpha \\ \mathrm{Cov}(Y_\alpha, Y_\beta) &= \begin{cases} \rho\sigma^2 & \text{if } \alpha \neq \beta \text{ in same row} \\ \tau\sigma^2 & \text{if } \alpha \neq \beta \text{ in same column} \\ 0 & \text{if } \alpha \neq \beta \text{ otherwise} \end{cases} \end{aligned}$$

with $0 \leq \rho \leq 1$ and $0 \leq \tau \leq 1$.

Pairwise variance in Example

B	D	G	A	F	C	E
A	G	C	D	F	B	E
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From $V_{BG} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{4}{5}\tau\right]$

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From $V_{BG} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{4}{5}\tau\right]$ to $V_{EF} = \frac{2\sigma^2}{5}\left[1-\rho+\tau\right]$

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$$\text{From} \quad V_{BG} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{4}{5}\tau\right] \quad \text{to} \quad V_{EF} = \frac{2\sigma^2}{5}\left[1-\rho+\tau\right]$$

$$\text{with average} \quad V = \frac{2\sigma^2}{5}\left[1-\rho-\frac{1}{15}\tau\right].$$

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- Simple to construct.
- Simple to randomize.
- Simple to analyse.
- Some treatment comparisons in some experiments will have a specially low or specially high variance,
  but the estimated variance is unbiased
  when averaged over all comparisons and
  all possible randomized plans.

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  but the estimate of that variance is higher.

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$$V = \frac{2\sigma^2}{r}\left[(1-\rho) - \frac{(r-1)\tau}{n-1}\right]$$

and

$$E(\hat{V}) = \frac{2E(M)}{r} = \frac{2\sigma^2}{r}\left[(1-\rho) + \frac{\tau}{n-1}\right]$$

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- Genuine treatment differences may not be detected.

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Easy to do this directly, eg

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Same bias in estimator of variance as for simple restricted randomization.

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- Analyse as usual.
- Same average variance as in randomized complete-block design, but with smaller range.
- The estimator of variance is unbiased
  when averaged over all comparisons in this one experiment.
- There is no separate estimate of ρ or τ,
  so treatments must be randomized
  and a single standard error given for all differences.

A design from the tables

A	B	C	D	E	F	G
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4. Subject to conditions (1)–(3), the spread of the variances of the estimators of simple treatment differences is as small as possible.

Pairwise variances in the example

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$$\text{Minimum} \quad V_{AD} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{2}{5}\tau\right]$$

$$\text{Maximum} \quad V_{AB} = \frac{2\sigma^2}{5}\left[1-\rho+\frac{2}{5}\tau\right]$$

$$\text{Average} \quad V = \frac{2\sigma^2}{5}(1-\rho)$$

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Average	$V = \frac{2\sigma^2}{5}(1-\rho)$	$\cdots -\frac{1}{15}\tau$	$\cdots -\frac{2}{3}\tau$
		one layout, normal method	simple restricted randomization

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- Unbiased estimator of the variance of every treatment contrast.
- There is no need to randomize treatments; the most important differences can be given the lowest variance.

Example of a row-column design

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$$
\begin{aligned}
V_{AB} &= 1.044 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \\
V_{AC} &= 1.089 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \\
V_{AD} &= 1.091 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \\
V &= 1.075 \times \frac{2}{5}(1-\rho-\tau)\sigma^2
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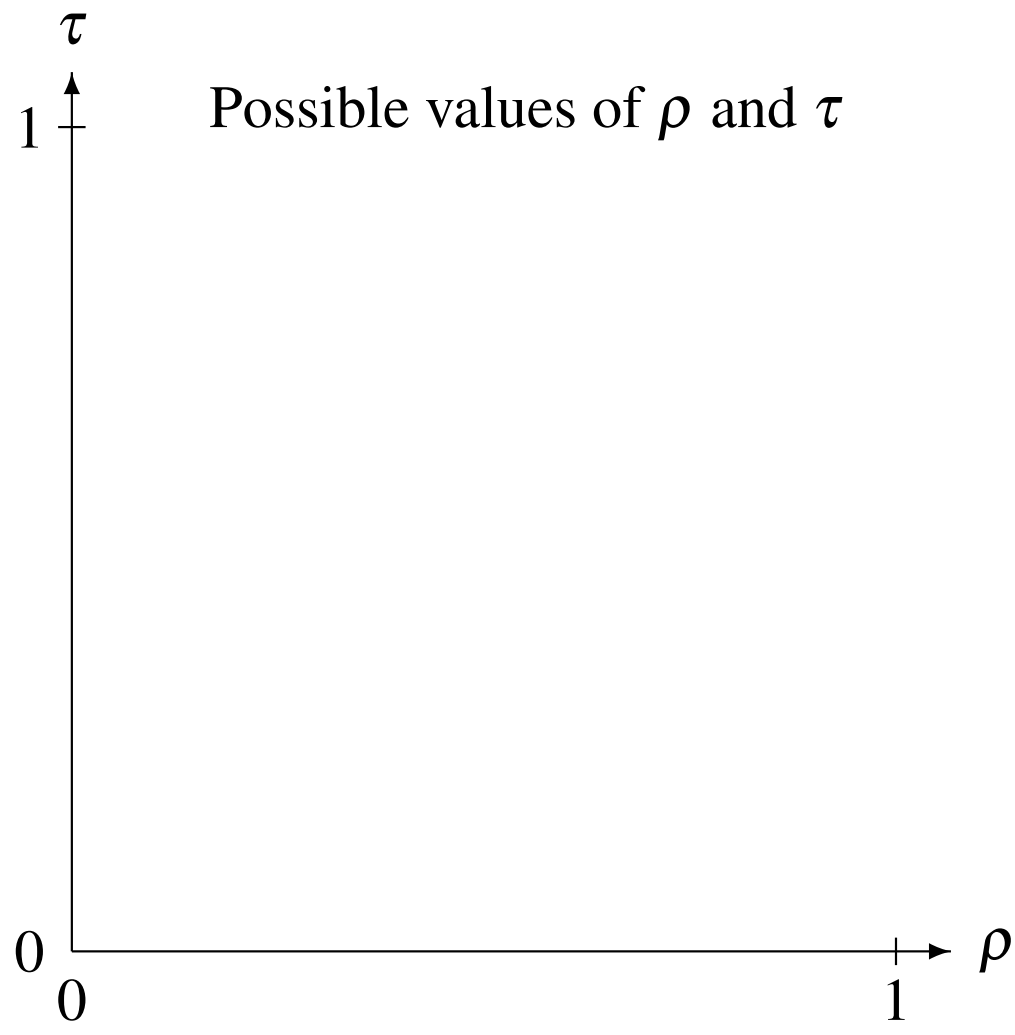
$$V_{AC} = 1.089 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \qquad V = \frac{2\sigma^2}{5}(1-\rho)$$

$$V_{AD} = 1.091 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \qquad \text{averaged over}$$

$$V = 1.075 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \qquad \text{randomizations}$$

Comparing super-valid restricted randomization and efficient row-column designs



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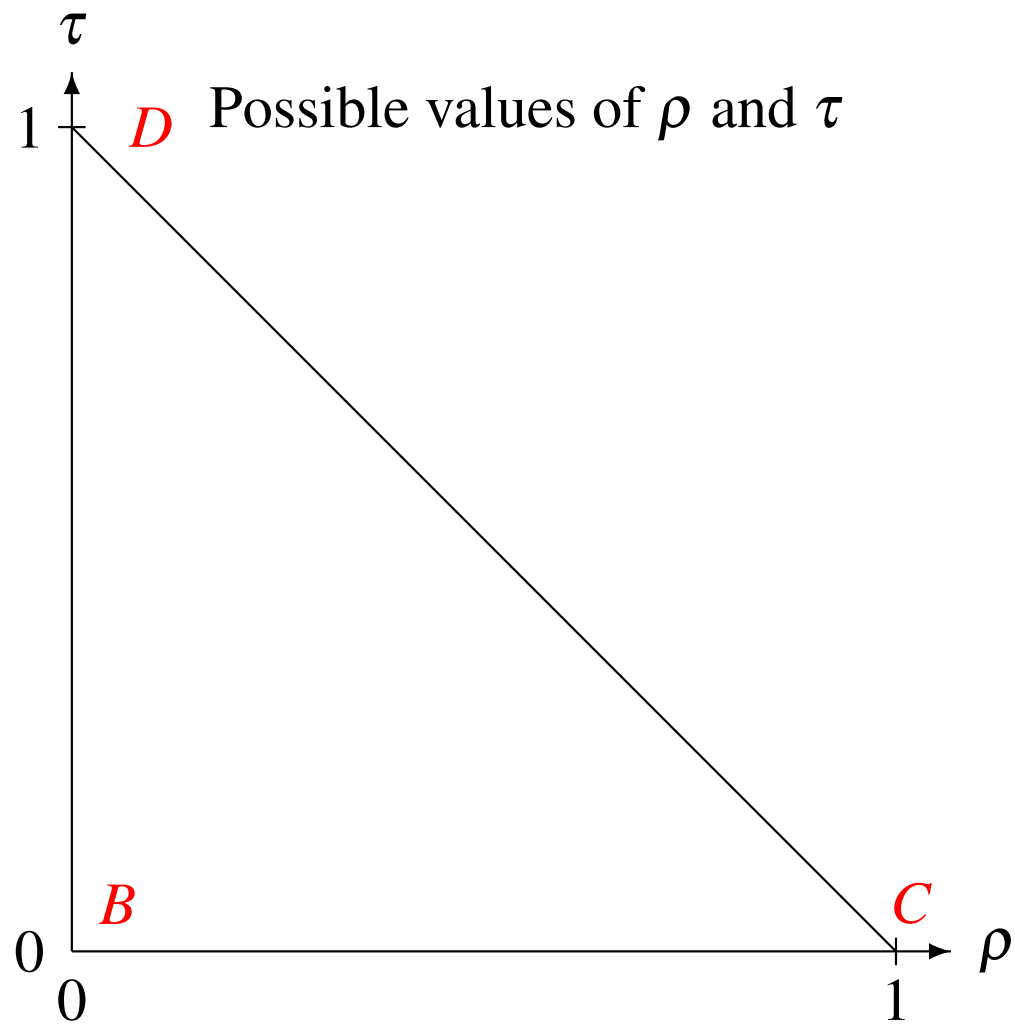
covariance matrix +ve definite
$\iff \rho + \tau < 1$



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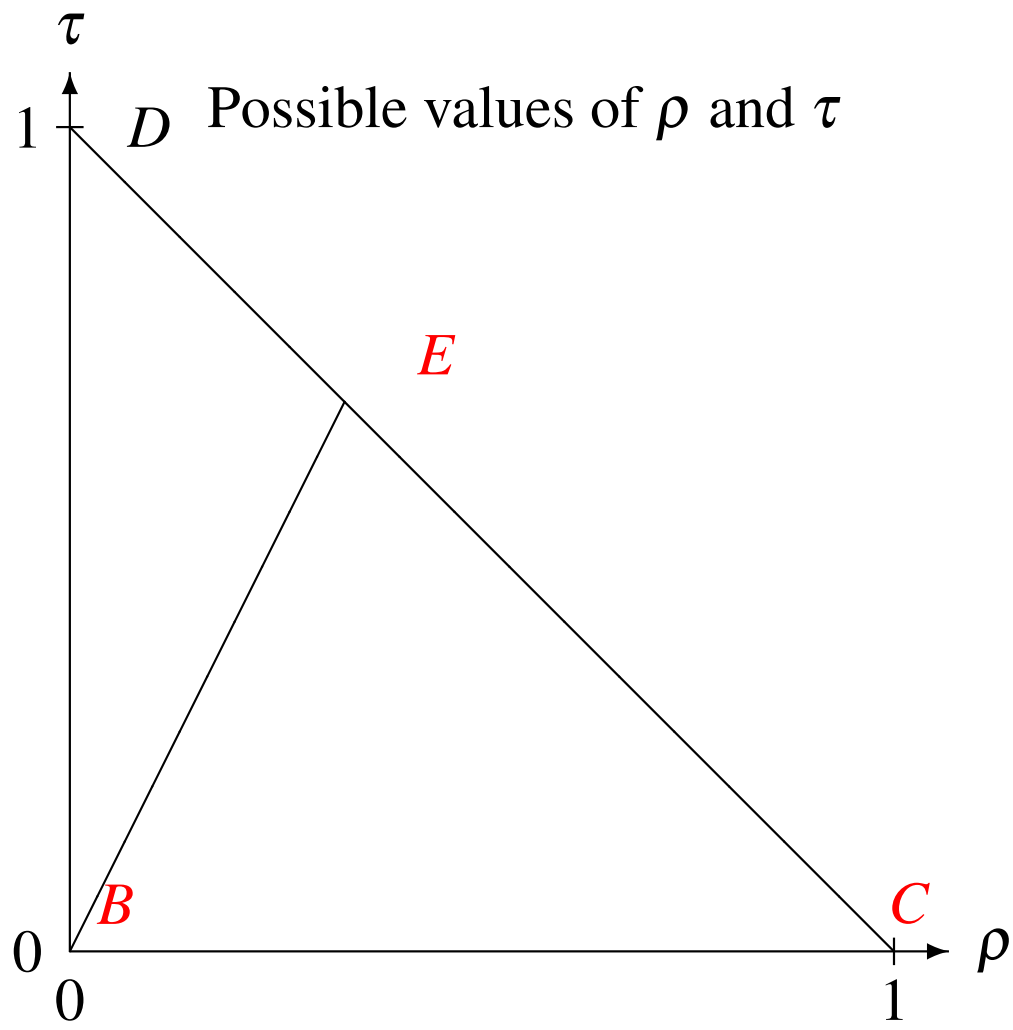

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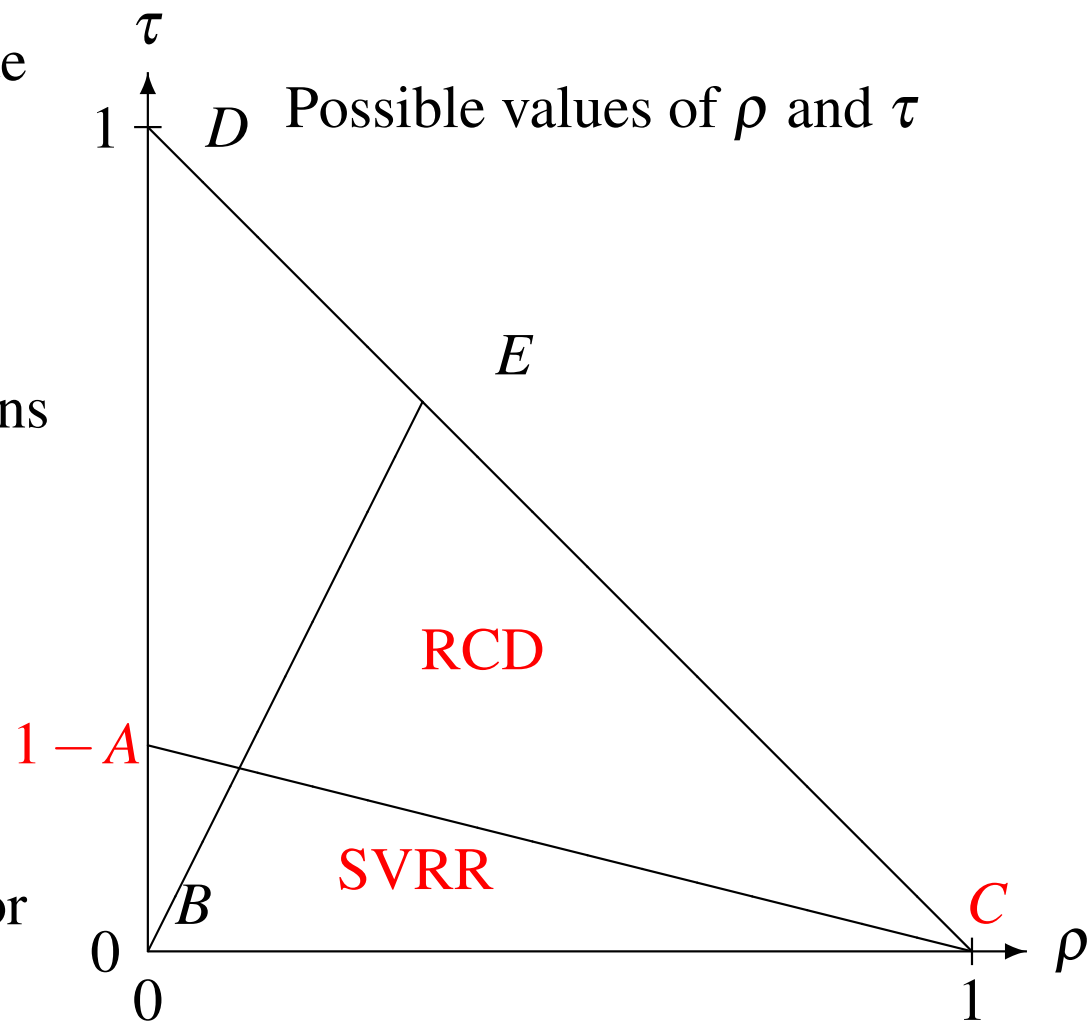
V in RCD $< V$ for SVRR
$\iff (1-A)(1-\rho) < \tau$
where A is the efficiency factor for the design in columns



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Hence unbiased estimators of $\sigma^2(1-\rho)$ and $\sigma^2\tau$ and of

$$V = \frac{2\sigma^2}{r} \left[(1-\rho) - \frac{(r-1)\tau}{n-1} \right].$$

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$$V = \frac{2\sigma^2}{r} \left[(1-\rho) - \frac{(r-1)\tau}{n-1} \right].$$

But this estimator of V does not have a χ^2 distribution,
so how do we do hypothesis tests?
Also, there are so few effective df for τ that
these estimates have very poor precision.