Optimal Designs for Diallel Experiments

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Sixth International Triennial Calcutta Symposium on Probability and Statistics
December 2006

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I shall state a result which gives optimality in some special cases and use this as a heuristic guide to optimality more generally.

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If we assume that the specific combining ability is zero, then the model is

$$\mathbf{E}\left(Y_{\{i,j\}}\right) = \alpha_i + \alpha_j$$

where α_i is the general combining ability for parental type i.

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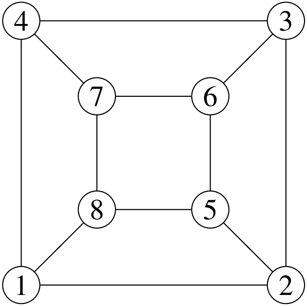
If there are n parental lines and the replication is r,
the design can be represented by a graph with n vertices and valency r:
each edge represents a cross that is used in the experiment.

Example of a design shown as a graph

The pairs are:
$\{1,2\}$, $\{1,4\}$, $\{1,8\}$, $\{2,3\}$, $\{2,5\}$, $\{3,4\}$ $\{3,6\}$, $\{4,7\}$, $\{5,6\}$, $\{5,8\}$, $\{6,7\}$, $\{7,8\}$.



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How should we choose which pairs to use?

What makes an incomplete-block design good? (1)

$$\begin{aligned} \text{Put} \quad \lambda_{ij} &= \text{number of blocks containing } i \text{ and } j \text{ if } i \neq j \\ \lambda_{ii} &= r \\ \Lambda &= [\lambda_{ij}]. \end{aligned}$$

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The matrix $I - \dfrac{1}{2r}\Lambda$ has eigenvalue 0 on the all-1 vector.

The other eigenvalues $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ are called the canonical efficiency factors in the within-blocks stratum.

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Put $A =$ harmonic mean of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$.

Then the average pairwise variance is $\frac{1}{rA}\sigma^2$.

What makes an incomplete-block design good? (2)

There are canonical efficiency factors

$$\begin{array}{lllll} \varepsilon_1, & \varepsilon_2, & \ldots, & \varepsilon_{n-1} & \text{in the within-blocks stratum} \\ \varepsilon_1^*, & \varepsilon_2^*, & \ldots, & \varepsilon_{n-1}^* & \text{in the between-blocks stratum} \end{array}$$

where $\varepsilon_i^* = 1 - \varepsilon_i$.

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For our analysis (using only block totals), an incomplete-block design is optimal if it maximizes

$$A^* = \text{harmonic mean of } \varepsilon_1^*,\ \varepsilon_2^*,\ \ldots,\ \varepsilon_{n-1}^*.$$

Good and bad news

Easy Theorem

$$
\begin{array}{ccc}
\text{balance} & & \\
\Updownarrow & & \\
\varepsilon_1 = \cdots = \varepsilon_{n-1} & \Longleftrightarrow & \varepsilon_1^* = \cdots = \varepsilon_{n-1}^* \\
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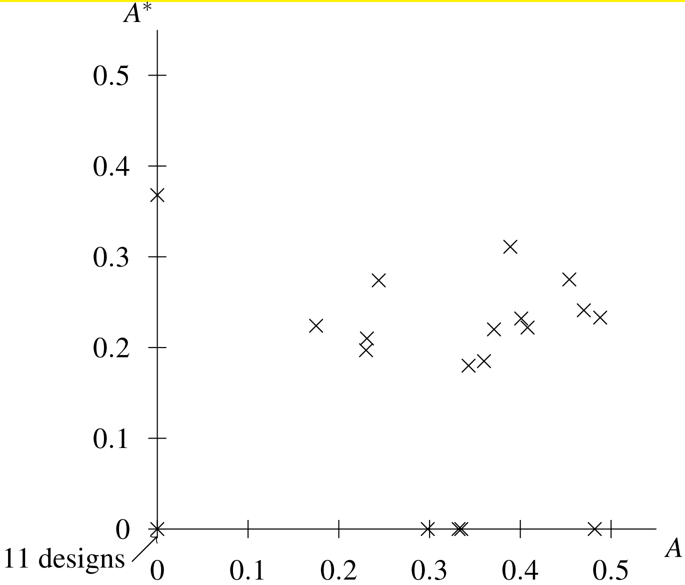
Warning!

If there is no balanced incomplete block design
(for the given numbers of blocks, treatments and block size),
then a design which is optimal for the usual analysis
may not be optimal for the analysis of block totals.

Curnow noticed this in 1963 when he examined some small designs.

All designs with $n = 8$ and $r = 3$



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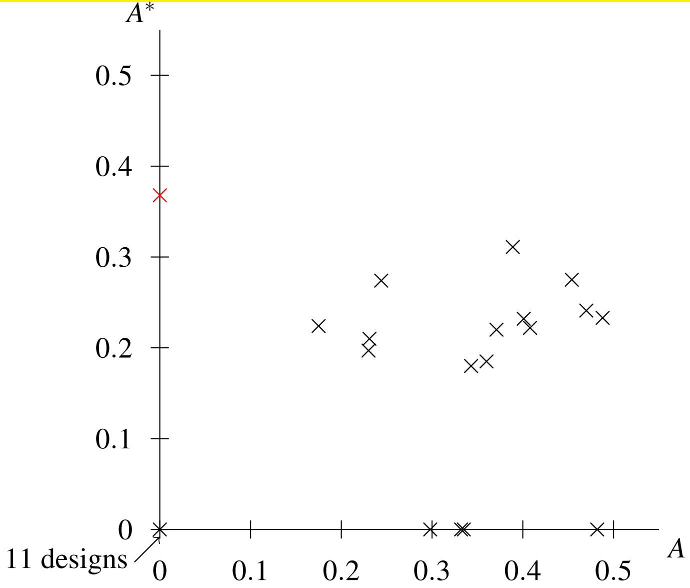
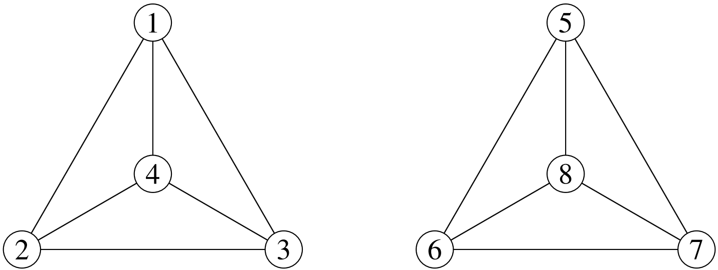


One extreme



The design is disconnected.

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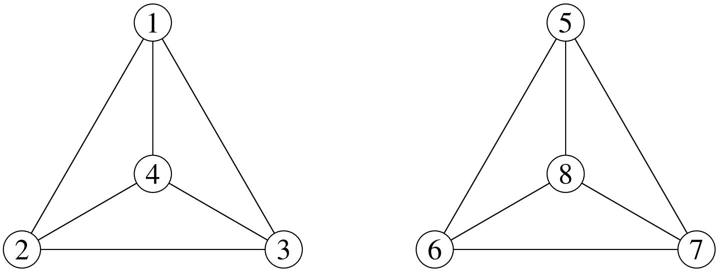

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The difference between the two components is not estimable within blocks.

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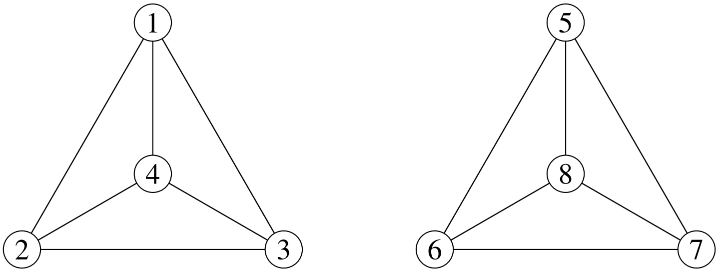

The design is disconnected.

The difference between the two components is not estimable within blocks.

But this design is optimal for the analysis using block totals.

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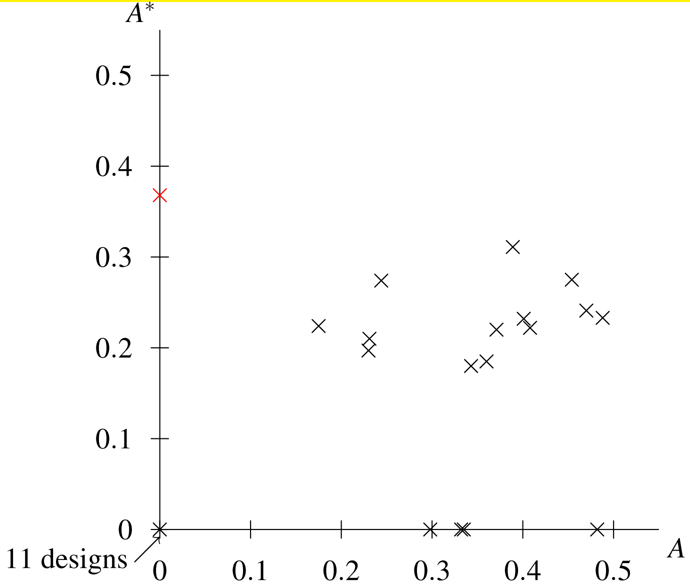

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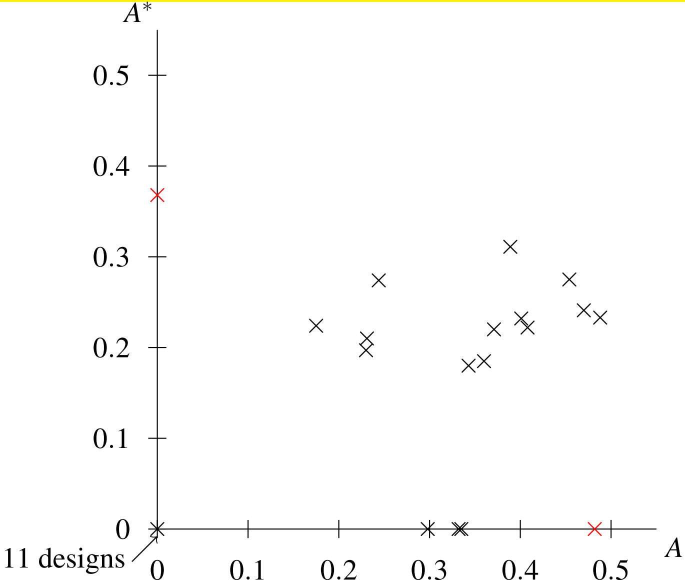
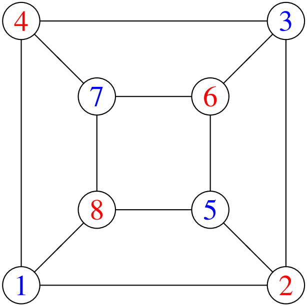


Almost the other extreme



The design is bipartite because every edge joins an even treatment to an odd treatment.

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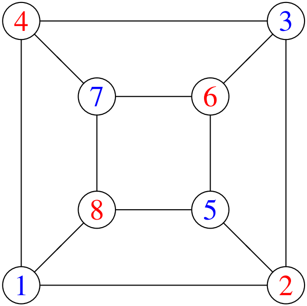

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The difference between even and odd is not estimable from the block totals.

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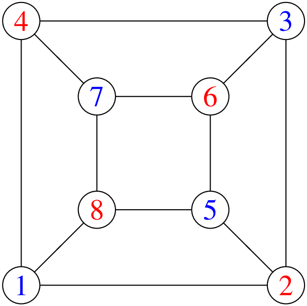

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The difference between even and odd is not estimable from the block totals.

But this design is 2nd best for the usual within-blocks analysis.

Theorem for incomplete-block designs with the usual analysis

In 1991, Cheng and Bailey proved that, for the usual analysis: if

- $\varepsilon_1, \ldots, \varepsilon_{n-1}$ take only two values, one of which is 1, and
- the design is partially balanced with two associate classes, and concurrences differing by 1,

then A is maximal.

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(The design is also E-optimal and D-optimal: in fact, all symmetric convex functions of the eigenvalues are maximized.)

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The same proof shows that, for the analysis using block totals,

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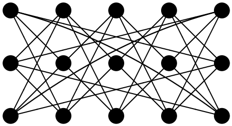
($E*$, $D*$ etc are also maximal.)

Mukerjee proved a similar result for E-optimality in 1997.

Example with $n = 15$ and $r = 4$



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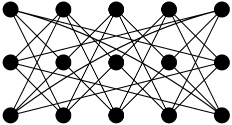

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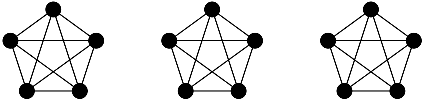

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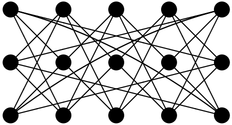

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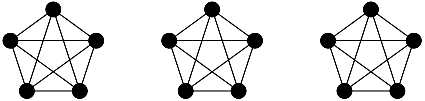

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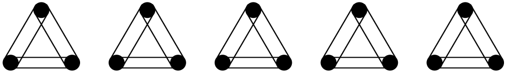

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Heuristic

For n parental lines, each involved in r crosses,
let q be the smallest number such that

- $r+1$ divides $n-q$

Then the optimal design should be of the following form:

- $\dfrac{n-q}{r+1}$ copies of the complete graph on $r+1$ vertices;

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Exception

For $r=2$ and $n=3m+10$, use m triangles and 2 pentagons.

Example for $n = 15$ and $r = 6$

The best design for the usual analysis is a triangular partially balanced design.

When used for an analysis by block totals, the pairwise variances are

0.4762	45 times
0.3810	60 times
0.4218	average

and $A^* = 0.3952$.

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The heuristic choice consists of one complete graph on 7 treatments and a graph on 8 points which has all edges except for diagonal pairs.

When used for an analysis by block totals, the pairwise variances are

0.4167	24 times
0.4000	21 times
0.3709	56 times
0.3333	4 times
0.3857	average

and $A^* = 0.4321$.

Tomatoes in glasshouses

Possible extensions

In experiments on tomatoes in glasshouses, several plants are grown close together. Each is tied to a vertical string. When the stalk reaches the top of the stake, it is tied around a horizontal rail. So several plants are tied around the same rail, and their fruit is all mixed up.

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I have always said that the plants which are mixed up like this should all be of the same variety. But perhaps we could have k different varieties: then we would measure a response whose expectation is

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However, the efficiency criteria are different for the two situations. If a balanced design exists it is optimal for both situations; otherwise, it can happen that the design that is most efficient for one situation is worst for the other. Thus conventional lists of optimal block designs must be treated with care.

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