Conflicts between optimality criteria for block designs with low replication

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Ongoing joint work with Alia Sajjad and Peter Cameron

What makes a block design good for experiments?

I have v treatments that I want to compare.
I have b blocks,
with space for k treatments (not necessarily distinct) in each block.
How should I choose a block design?

Two designs with $v = 15$, $b = 7$, $k = 3$: which is better?

1	1	2	3	4	5	6
2	4	5	6	10	11	12
3	7	8	9	13	14	15

replications differ by ≤ 1

1	1	1	1	1	1	1
2	4	6	8	10	12	14
3	5	7	9	11	13	15

queen-bee design

The replication of a treatment is its number of occurrences.

A design is a queen-bee design if there is a treatment that occurs in every block.

Two designs with $v = 7$, $b = 7$, $k = 3$: which is better?

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

balanced (2-design)

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2

non-balanced

A binary design is balanced if every pair of distinct treaments occurs together in the same number of blocks.

Concurrence graphs of two designs: $v = 15$, $b = 7$, $k = 3$

1	1	2	3	4	5	6
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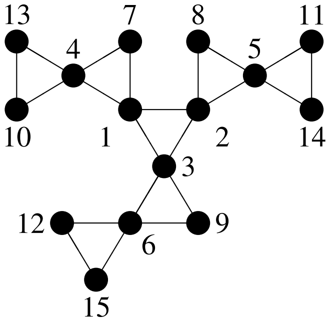

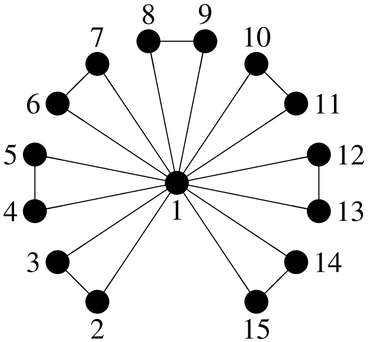

Estimation and variance

We measure the response Y on each unit in each block.

If that unit has treatment i and block m, then we assume that

$$Y = \tau_i + \beta_m + \text{random noise.}$$

We want to estimate contrasts $\sum_i x_i \tau_i$ with $\sum_i x_i = 0$.

In particular, we want to estimate all the simple differences $\tau_i - \tau_j$.

Put $V_{ij} =$ variance of the best linear unbiased estimator for $\tau_i - \tau_j$.

We want all the V_{ij} to be small.

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The design is called

- A-optimal if it minimizes
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over all block designs for v treatments in b blocks of size k.

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- A-optimal if it minimizes
  the average of the pairwise variances V_{ij};
- D-optimal if it minimizes
  the volume of the confidence ellipsoid for $(\tau_1, \ldots, \tau_v)$,
  subject to $\sum \tau_i = 0$;

over all block designs for v treatments in b blocks of size k.

Balanced designs are optimal

Theorem

If there is a balanced incomplete-block design (BIBD) for v treatments in b blocks of size k, then it is A, D and E-optimal.

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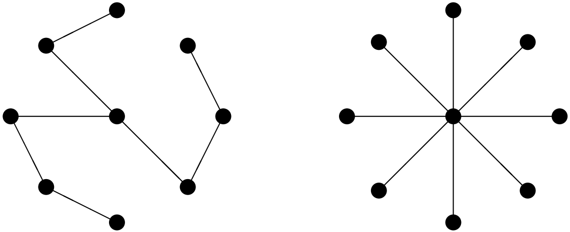
Hence a general idea that

- designs optimal on any of these criteria
  should be close to balanced and close to equi-replicate
- designs optimal on one of these criteria
  are not very bad on either of the others.

Block size 2: least replication

If $k = 2$ then the design is the same as its concurrence graph, and connectivity requires $b \geq v - 1$.

If $b = v - 1$ then all connected designs are trees.



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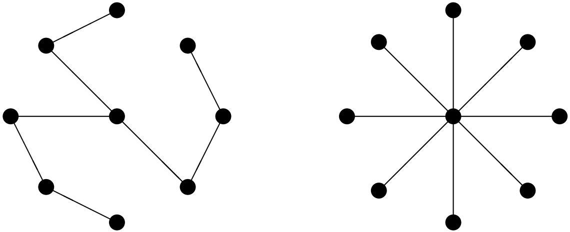

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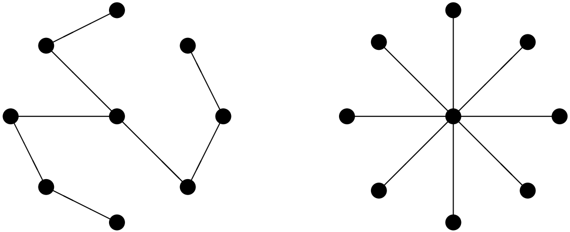
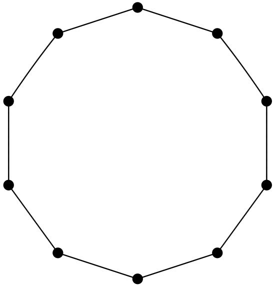


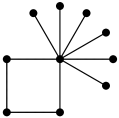
The D-criterion does not differentiate them.

The only A-optimal designs are the stars.

Block size 2: one more block: D

If $k = 2$ and $b = v$ then the design consists of a cycle with trees attached to some vertices.





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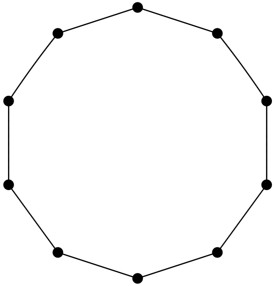

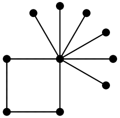

The cycle is uniquely D-optimal when $b = v$.

Block size 2: one more block

D-optimal designs	cycle	always
A-optimal designs	cycle	if $v \leq 8$
	square with leaves attached	if $9 \leq v \leq 12$
	triangle with leaves attached	if $12 \leq v$

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and the A-optimal designs are far from equi-replicate.
The change is sudden, not gradual.

Block size 3, but minimal b

The remaining arguments extend easily to general block size.

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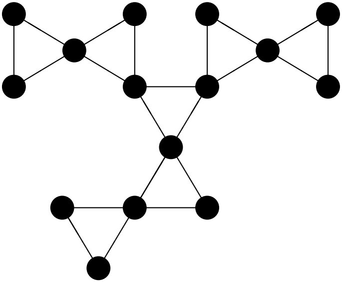
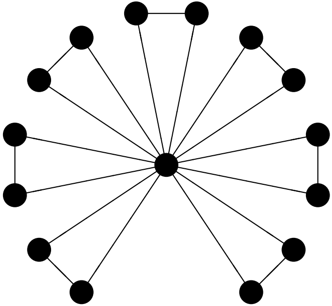
When $k = 3$, for a connected design, we need $2b \geq v - 1$.

If $2b + 1 = v$ then all designs are gum-trees, in the sense that there is a unique sequence of blocks from any one treatment to another.





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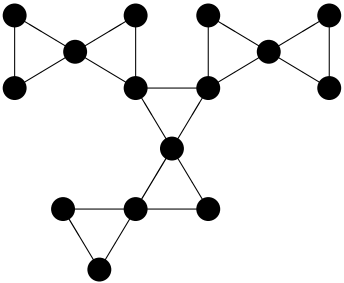

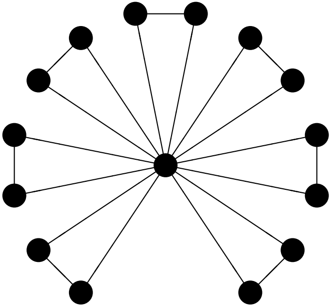

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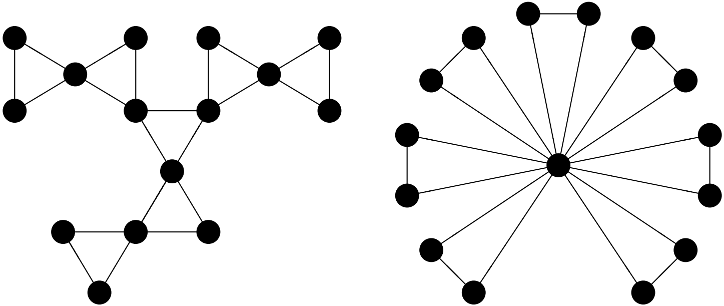

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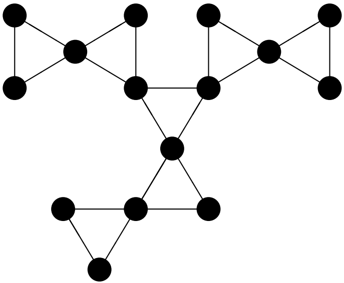
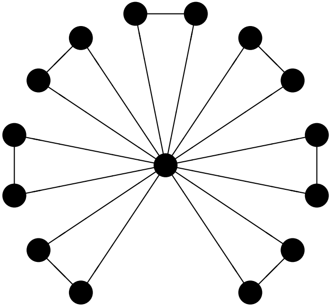


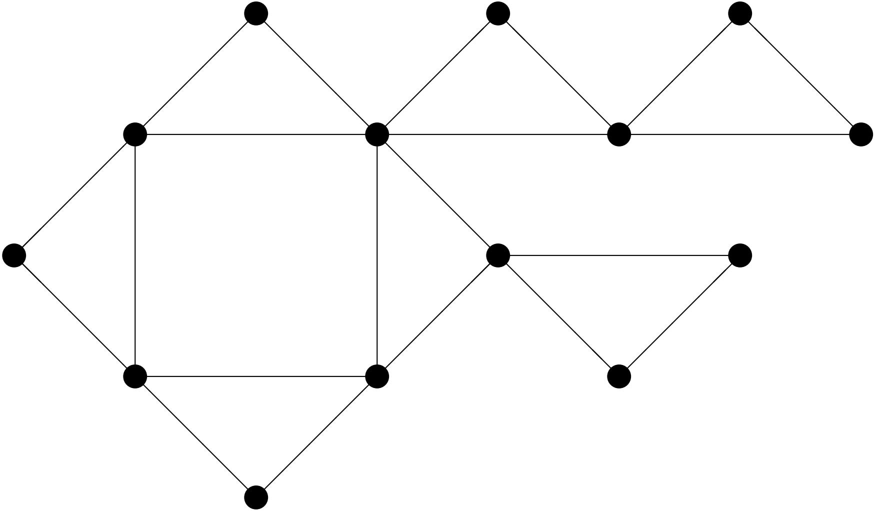


The D-criterion does not differentiate them.

The only A-optimal designs are the queen-bee designs.

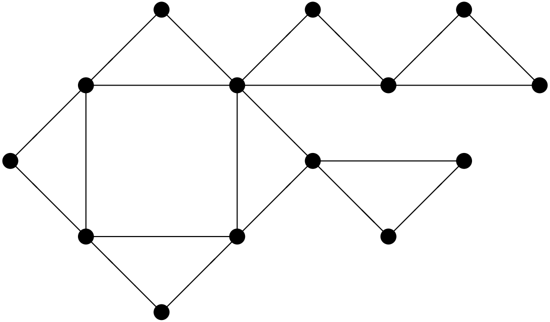
Block size 3, but $b =$ minimal $+1$

If $2b = v$ then G is a gum-cycle with gum-trees attached.



Block size 3, but $b =$ minimal $+1$: D

Suppose that there are s blocks in the gum-cycle.



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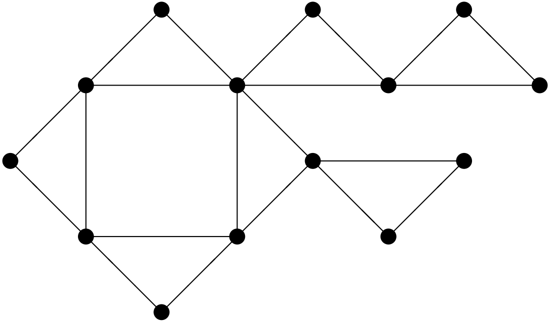


The design is D-optimal if and only if $s = b$.

Block size k, but $b =$ minimal $+1$: D-optimality

This argument extends to all block sizes.

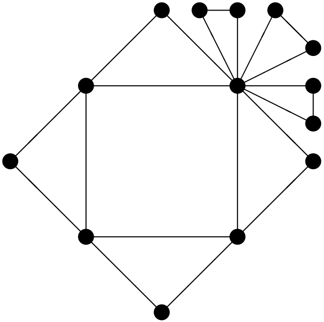
If $v = b(k-1)$ then the only D-optimal designs are the gum-cycles.

Block size 3, but $b =$ minimal $+1$: A

Suppose that there are s blocks in the gum-cycle.

Block size 3, but $b =$ minimal $+1$: A

Suppose that there are s blocks in the gum-cycle.
Then the only candidate for A-optimality consists of $b - s$ triangles attached to a central vertex of the gum-cycle.



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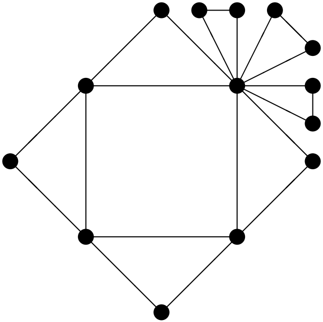

A-optimal designs do not have s large.

Block size k, but $b =$ minimal $+1$: A-optimality

This argument extends to all block sizes.

If $v = b(k-1)$ then the only A-optimal designs consist of
a gum-cycle of s_0 blocks together with
$b - s_0$ blocks attached to a central vertex of the gum-cycle.

The value of s_0 depends on b and k, but it is never large.