Rectangular experiments: restricted randomization or row-column designs?

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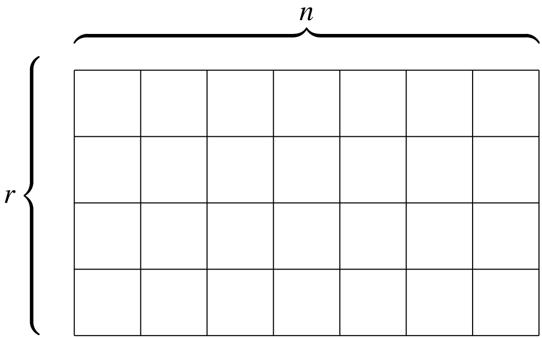
The problem

An agricultural experiment to compare n treatments.
The experimental area has r rows and n columns.



Use a randomized complete-block design with rows as blocks.
(In each row, choose one of the $n!$ orders with equal probability.)

What should we do if the randomization produces a plan with one treatment always at one side of the rectangle?

Example

Federerer (1955 book): guayule trees

B	D	G	A	F	C	E
A	G	C	D	F	B	E
G	E	D	F	B	C	A
B	A	C	F	G	E	D
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Solution: Use a carefully chosen Latinized design; REML/ANOVA estimates of variance components

Assumed model

Y_α is the response on plot α.

$E(Y_\alpha) = \theta_i$ where i is the treatment on α.

$$\begin{aligned} \mathrm{Var}(Y_\alpha) &= \sigma^2 \qquad \text{for all } \alpha \\ \mathrm{Cov}(Y_\alpha, Y_\beta) &= \begin{cases} \rho\sigma^2 & \text{if } \alpha \neq \beta \text{ in same row} \\ \tau\sigma^2 & \text{if } \alpha \neq \beta \text{ in same column} \\ 0 & \text{if } \alpha \neq \beta \text{ otherwise} \end{cases} \end{aligned}$$

with $0 \leq \rho \leq 1$ and $0 \leq \tau \leq 1$.

Concurrence

$\lambda_{ij} =$ number of pairs of plots in the same column getting treatments i and j.

B	D	G	A	F	C	E
A	G	C	D	F	B	E
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$$\lambda_{AD} = 0+1+0+1+0+0+1 = 3$$

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$$\begin{aligned}\lambda_{AD} &= 0+1+0+1+0+0+1 = & 3\\ \lambda_{AB} &= 2+1+0+0+0+1+0 = & 4\end{aligned}$$

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$$
\begin{aligned}
\lambda_{AD} &= 0+1+0+1+0+0+1 = & 3 \\
\lambda_{AB} &= 2+1+0+0+0+1+0 = & 4 \\
\lambda_{AA} &= 1+1+0+1+0+1+1 = & 5
\end{aligned}
$$

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$\lambda_{ij} =$ number of pairs of plots in the same column getting treatments i and j.

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A	G	C	D	F	B	E
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\lambda_{AA} &= 1+1+0+1+0+1+1 = & 5 \\
\lambda_{BB} &= 4+1+0+0+1+1+0 = & 7
\end{aligned}$$

Pairwise variance

$$
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\begin{aligned}
V_{ij} &= \text{variance of the estimator of } \theta_i - \theta_j \\
&= \frac{\sigma^2}{r^2}\left[2r - 2r\rho + (\lambda_{ii} - r)\tau + (\lambda_{jj} - r)\tau - 2\lambda_{ij}\tau\right]
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= \frac{\sigma^2}{r^2}\left[2r(1-\rho) + (\lambda_{ii} + \lambda_{jj} - 2\lambda_{ij} - 2r)\tau\right]
$$

Pairwise variance in the example

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From $\quad V_{BG} = \frac{2\sigma^2}{5}\left[1 - \rho - \frac{4}{5}\tau\right]$

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From $\quad V_{BG} = \frac{2\sigma^2}{5}\left[1 - \rho - \frac{4}{5}\tau\right] \quad$ to $\quad V_{EF} = \frac{2\sigma^2}{5}[1 - \rho + \tau]$

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with average $\quad V = \dfrac{2\sigma^2}{5}\left[1-\rho-\dfrac{1}{15}\tau\right].$

Calculations

Put $\quad V = \dfrac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} V_{ij} \quad$ and put $\quad D = \sum_{i=1}^{n} \lambda_{ii}.$

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so smaller $V \implies$ larger $\hat{V}$.

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- Simple to construct.
- Simple to randomize.
- Simple to analyse.
- Some treatment comparisons in some experiments will have a specially low or specially high variance,
  but the estimated variance is unbiased
  when averaged over all comparisons and
  all possible randomized plans.

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$$V = \frac{2\sigma^2}{r}\left[(1-\rho) - \frac{(r-1)\tau}{n-1}\right]$$

and

$$E(\hat{V}) = \frac{2E(M)}{r} = \frac{2\sigma^2}{r}\left[(1-\rho) + \frac{\tau}{n-1}\right]$$

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- Genuine treatment differences may not be detected.

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Easy to do this directly, eg

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Same bias in estimator of variance as for simple restricted randomization.

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- Needs tables of designs.
- Randomize rows, columns and treatments.
- Analyse as usual.
- Same average variance as in randomized complete-block design, but with smaller range.
- The estimator of variance is unbiased when averaged over all comparisons in this one experiment.
- There is no separate estimate of ρ or τ, so treatments must be randomized and a single standard error given for all differences.

A design from the tables

A	B	C	D	E	F	G
D	E	F	C	A	B	G
A	G	F	B	C	E	D
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1. In every pair of rows, there is exactly one column in which the two treatments are the same.

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3. If $m_i =$ the number of columns in which treatment i occurs twice, then $m_i - m_j \in \{-1, 0, 1\}$ for all other treatments j.
4. Subject to conditions (1)–(3), the spread of the variances of the estimators of simple treatment differences is as small as possible.

Pairwise variances in the design from the tables

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$$\begin{array}{lll}
\text{Minimum} & V_{AD} & = \dfrac{2\sigma^2}{5}\left[1-\rho-\dfrac{2}{5}\tau\right] \\
\text{Maximum} & V_{AB} & = \dfrac{2\sigma^2}{5}\left[1-\rho+\dfrac{2}{5}\tau\right] \\
\text{Average} & V & = \dfrac{2\sigma^2}{5}(1-\rho)
\end{array}$$

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Minimum $V_{AD} = \frac{2\sigma^2}{5}\left[1-\rho-\frac{2}{5}\tau\right]$ $\cdots-\frac{4}{5}\tau$

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Average $V = \frac{2\sigma^2}{5}(1-\rho)$ $\cdots-\frac{1}{15}\tau$

one layout,
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Average $V = \frac{2\sigma^2}{5}(1 - \rho)$ $\quad \cdots - \frac{1}{15}\tau$ $\quad \cdots - \frac{2}{3}\tau$

one layout, normal method

simple restricted randomization

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- Average variance may be less than, or more than,
  the average variance in randomized complete-block design,
  depending on the size of the correlations.
- Unbiased estimator of the variance of every treatment contrast.
- There is no need to randomize treatments; the most important differences can be given the lowest variance.

Columns form an Incomplete-block design (IBD)

Given an incomplete-block design for
n treatments in n blocks of size r,
define the number A $(0 < A < 1)$, depending on the design, by

$$A = \frac{2\sigma^2}{rV}$$

if the analysis uses information orthogonal to blocks.

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Hall's Marriage Theorem $\Longrightarrow$ the blocks of this IBD can be arranged as the columns of a row-column design so that each treatment occurs once in each row.

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define the number A $(0 < A < 1)$, depending on the design, by

$$A = \frac{2\sigma^2}{rV}$$

if the analysis uses information orthogonal to blocks.

Choose the optimal IBD: the one with the largest value of A.

Hall's Marriage Theorem $\Longrightarrow$ the blocks of this IBD can be arranged as the columns of a row-column design so that each treatment occurs once in each row.

Randomize rows and columns.

Example of a row-column design

A	B	C	D	E	F	G
B	C	D	E	F	G	A
C	D	E	F	G	A	B
D	E	F	G	A	B	C
E	F	G	A	B	C	D

$$
\begin{aligned}
V_{AB} &= 1.044 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \\
V_{AC} &= 1.089 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \\
V_{AD} &= 1.091 \times \frac{2}{5}(1-\rho-\tau)\sigma^2 \\
V &= 1.075 \times \frac{2}{5}(1-\rho-\tau)\sigma^2
\end{aligned}
$$

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$$V_{AB} = 1.044 \times \frac{2}{5}(1-\rho-\tau)\sigma^2$$

normal method

$$V_{AC} = 1.089 \times \frac{2}{5}(1-\rho-\tau)\sigma^2$$

$$V = \frac{2\sigma^2}{5}(1-\rho)$$

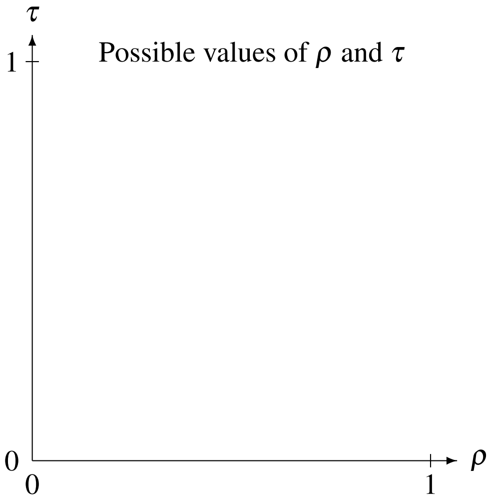
$$V_{AD} = 1.091 \times \frac{2}{5}(1-\rho-\tau)\sigma^2$$

averaged over

$$V = 1.075 \times \frac{2}{5}(1-\rho-\tau)\sigma^2$$

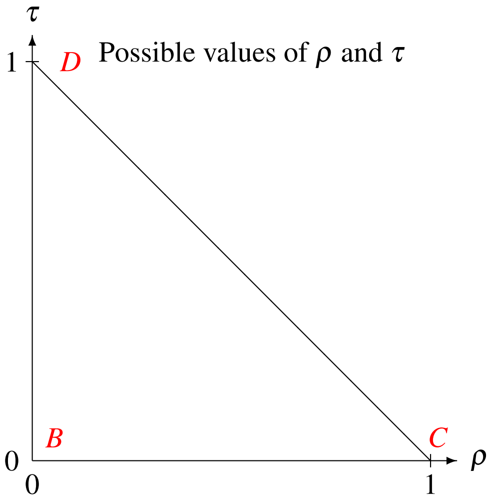
randomizations

Comparing super-valid restricted randomization and efficient row-column designs



Comparing super-valid restricted randomization and efficient row-column designs

covariance matrix +ve definite
$\iff \rho + \tau < 1$



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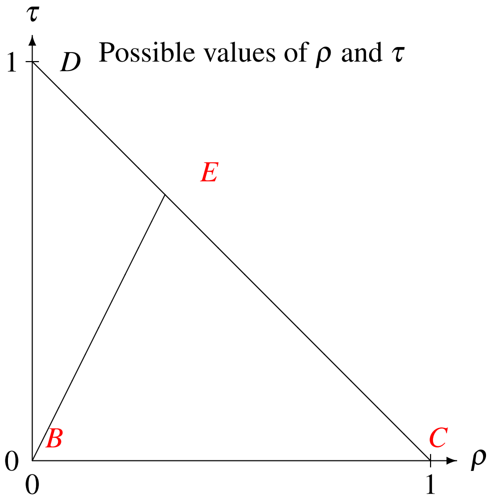
$\iff n\rho > r\tau$

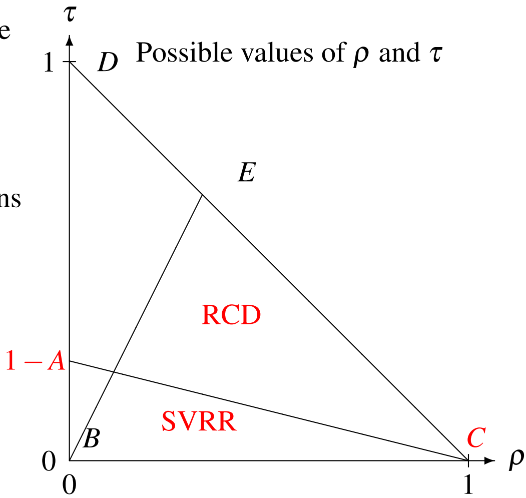


Comparing super-valid restricted randomization and efficient row-column designs

covariance matrix +ve definite
$\iff \rho + \tau < 1$

rows more variable than columns
$\iff n\rho > r\tau$

V in RCD $<$ V for SVRR
$\iff (1-A)(1-\rho) < \tau$



Use a carefully chosen Latinized design with REML/ANOVA estimates of variance components

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Randomize rows and columns.

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Hence unbiased estimators of $\sigma^2(1-\rho)$ and $\sigma^2\tau$ and of

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But this estimator of V does not have a χ^2 distribution,
so how do we do hypothesis tests?
Also, there are so few effective df for τ that
these estimates have very poor precision.