Block designs, spanning trees and resistance in electrical networks

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Abstract

The last decade has seen a huge number of microarray experiments performed. If we ignore some complications, the designs for these experiments are just block designs with block size two, so they can also be regarded as graphs, possibly with multiple edges.
Statisticians usually rate block designs by (at least) two criteria. When the block size is two, a design is D-optimal if it has the maximum number of spanning trees, and it is A-optimal if it has the minimum total of pairwise resistances when there is a unit resistance on each edge (block). Experience with block designs in other situations suggests that these two criteria agree closely at the top end. However, microarray experiments are usually done with such a small number b of blocks, relative to the number t of treatments, that the two criteria give opposite ranks.
I shall show that, if b is too small relative to t, this happens for all sufficiently large t.

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$$v \text{ treatments} \longrightarrow v \text{ vertices}$$
$$b \text{ blocks} \longrightarrow b \text{ edges}$$

Designs for 6 treatments (6 vertices)

12 blocks (edges)

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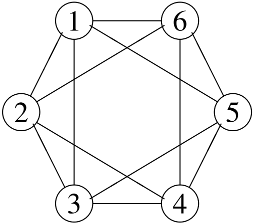

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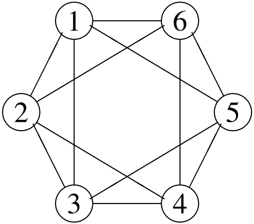

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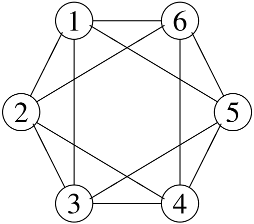

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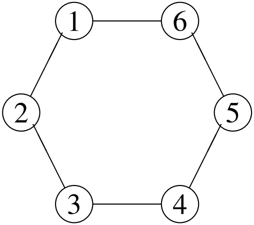

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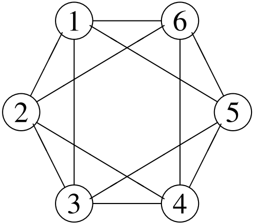

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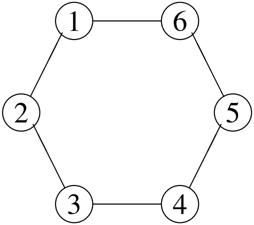

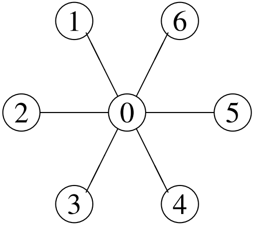

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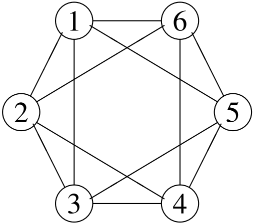

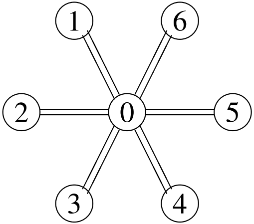

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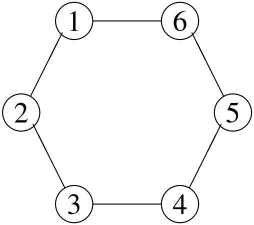

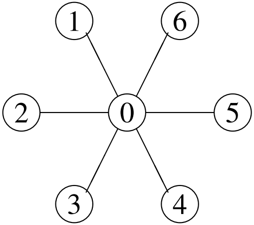

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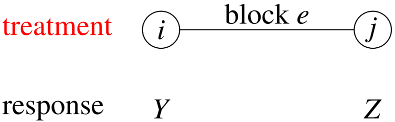

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treatment

response $\quad Y \qquad Z$

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Put $V_{ij} =$ variance of the estimator for $\tau_i - \tau_j$ using the whole graph.

We want all the V_{ij} to be small.

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  —equivalently, it minimizes the volume of the confidence ellipsoid for $(\tau_1, \ldots, \tau_v)$

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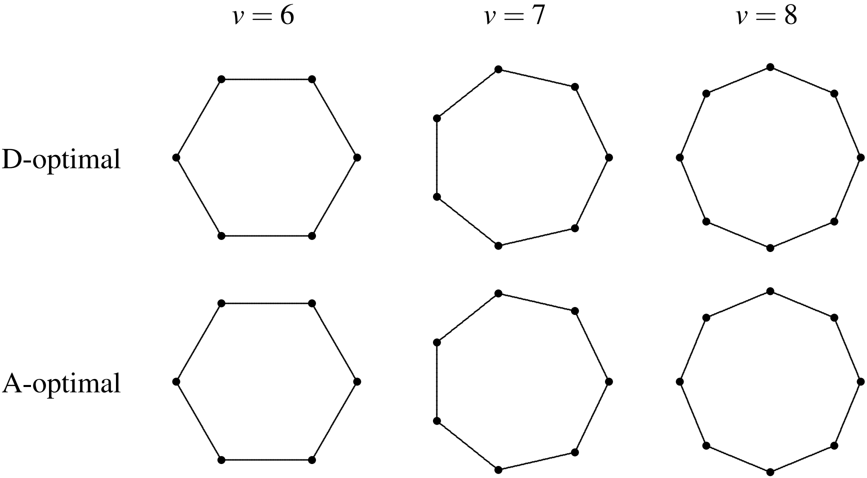
What happens when $b = v$?

Computer investigation by

- Jones and Eccleston (1980)
- Kerr and Churchill (2001)
- Wit, Nobile and Khanin (2005)
- Ceraudo (2005).

Optimal designs when $b = v$



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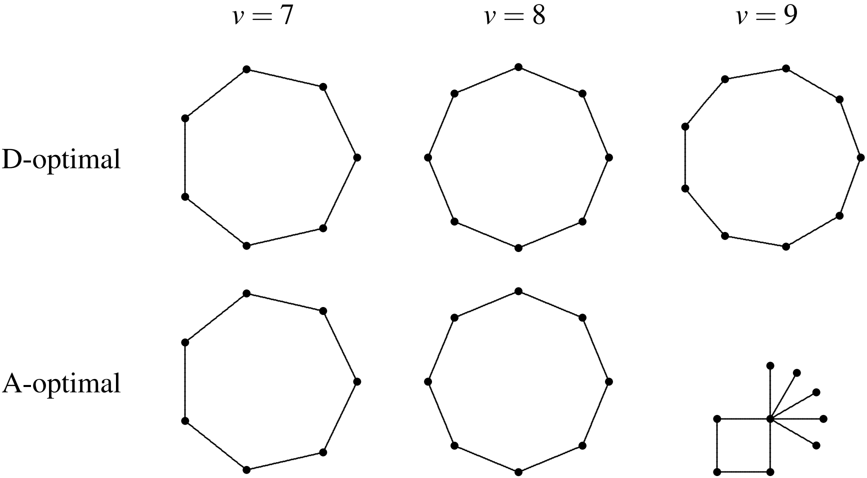

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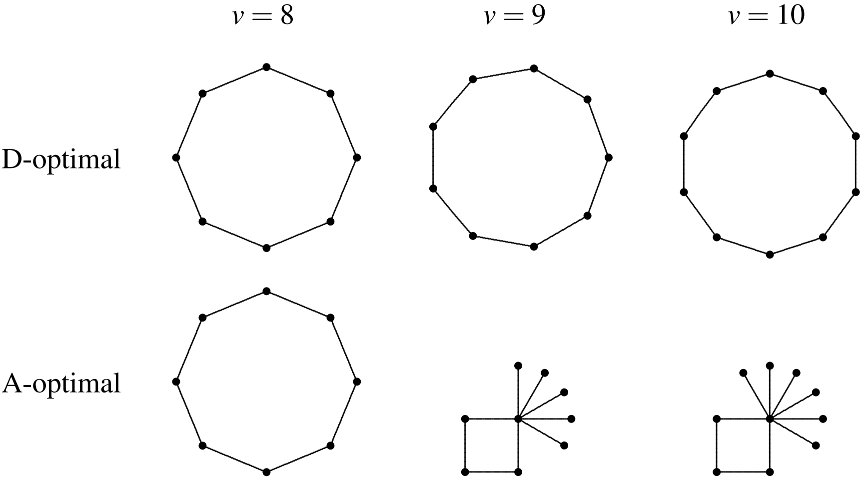

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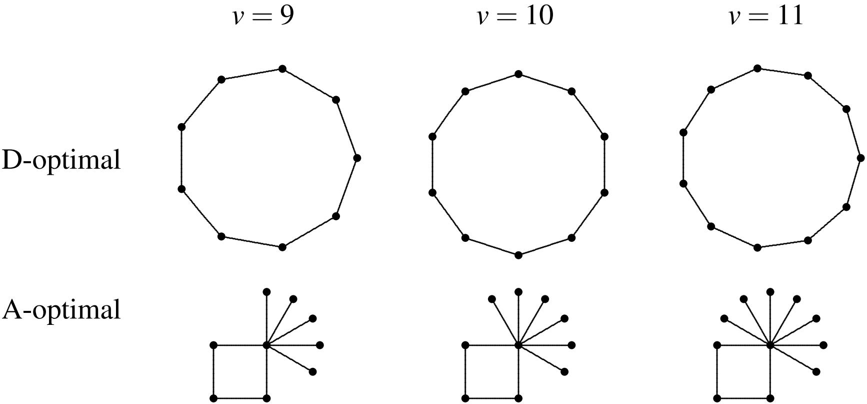

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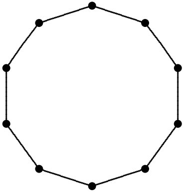
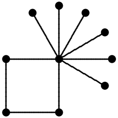


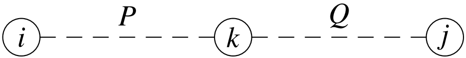
10 spanning trees



4 spanning trees

The loop design is uniquely D-optimal when $b = v$.

A-optimality: series



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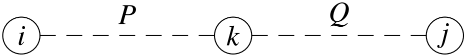

In our block design,

$$\begin{aligned} P &= \text{estimator for } \tau_i - \tau_k \text{ with variance } d \\ Q &= \text{estimator for } \tau_k - \tau_j \text{ with variance } e \end{aligned}$$

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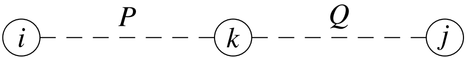

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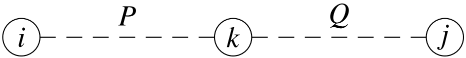

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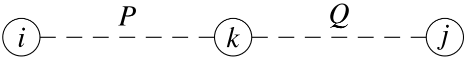

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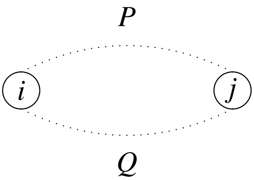

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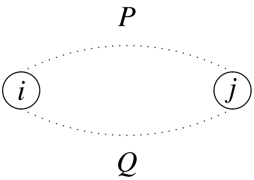

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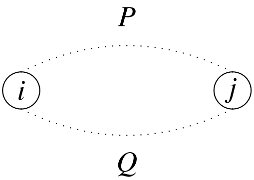

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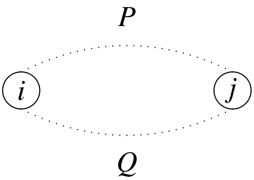

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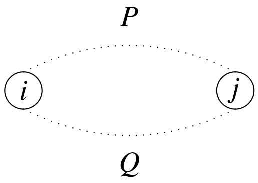

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Eureka!

Variance is the same as resistance!

Block design $\longrightarrow$ (multi-)graph

$\longrightarrow$ electrical network with resistance 1 in each edge

Use Ohm's Law and Kirchhoff's Laws to calculate each variance V_{ij} algebraically.

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The A-optimal design with v treatments (vertices) and v blocks (edges) is

- *the loop design (single circuit) if* $v \leq 8$;
- *a star glued to a quadrilateral if* $9 \leq v \leq 11$;
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Bad news theorem

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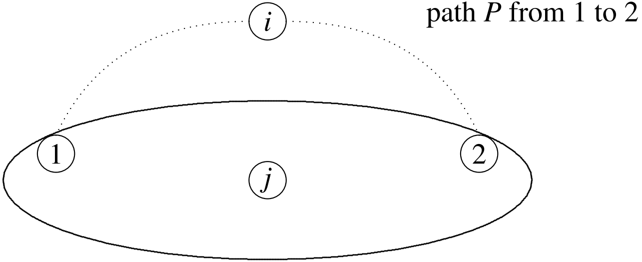
D-optimal designs do not have leaves.

Proof outline



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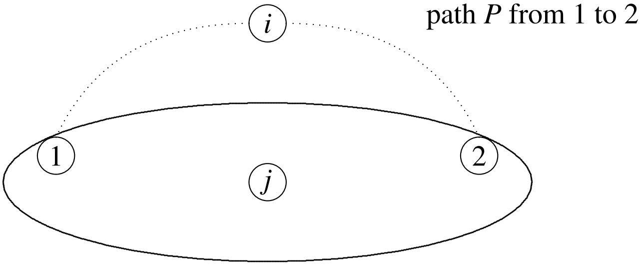

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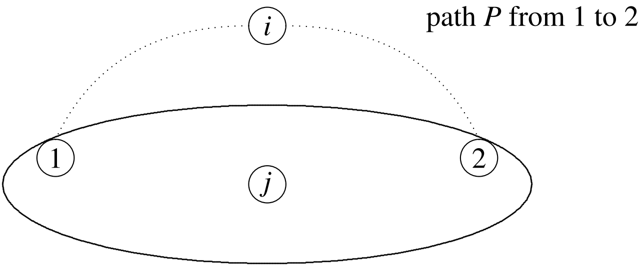

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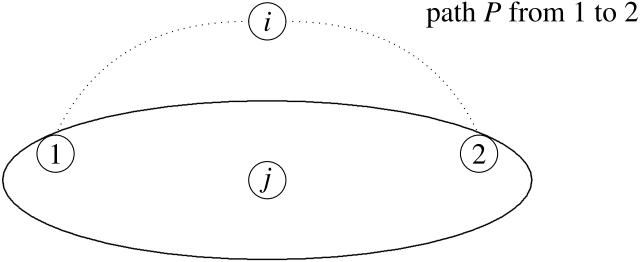

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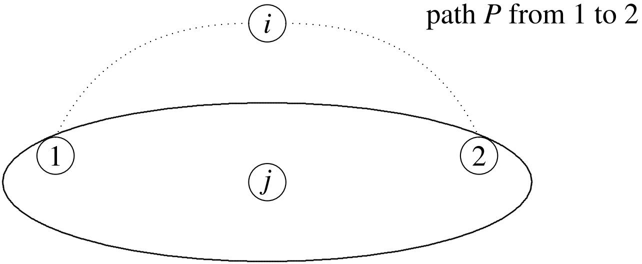
4. Large enough $v \implies$ large enough $n \implies \underset{\text{all } i,\, j}{\text{Average}}\, V_{ij} \geq 2$.

Proof outline



1. $\underset{i \in P}{\text{Average}}\, V_{ij} \geq \frac{1}{2}(V_{1j} + V_{2j})$ for all j in $G \setminus P$.
2. $\underset{\text{all } i,\, j}{\text{Average}}\, V_{ij} \geq \underset{i,\, j \in G_0}{\text{Average}}\, V_{ij}$
3. Inserting $n - 1$ vertices into all edges of G_0 multiplies all original variances by n.
4. Large enough $v \implies$ large enough $n \implies \underset{\text{all } i,\, j}{\text{Average}}\, V_{ij} \geq 2$.
5. In the star $K_{1,v-1}$, we have $\underset{\text{all } i,\, j}{\text{Average}}\, V_{ij} = \dfrac{2(v-1)}{v} \leq 2$.