Queen Mary
University of London

MAS 108 — Probability I

In-term Test — **10 November 2004, 12:10pm–12:55pm**

Write your name and student number in the spaces below.

Answer all questions. Write all your answers in the boxes provided.

Electronic calculators may not be used in this examination.

Name: ______________________________

Student Number: ______________________________

1 (15 marks) On a dry day, the probability that my bus is on time is 0.8, but on a wet day, the probability is 0.4. According to the weather forecast, the chance of rain tomorrow is 75%. What is the probability that my bus will be on time tomorrow? (You should explain your working.)

Let $A =$ 'rain tomorrow', $B =$ 'bus is on time'. Then A and A' form a partition, with

$$P(A) = \frac{3}{4}, \quad P(A') = \frac{1}{4}, \quad P(B \mid A) = \frac{2}{5}, \quad P(B \mid A') = \frac{4}{5}.$$

By the Theorem of Total Probability,

$$P(B) = P(B \mid A)P(A) + P(B \mid A')P(A') = \frac{2}{5} \times \frac{3}{4} + \frac{4}{5} \times \frac{1}{4} = \frac{1}{2}.$$

2 (15 marks) Five cards bearing the numbers 1 to 5 are put in a box. I draw a card and note its number, but do not replace it in the box. Then I draw another card and note its number.

(a) Write down the sample space. (5 marks)

$$\mathscr{S} = \{12, 13, 14, 15, 21, 23, 24, 25, 31, 32, 34, 35, 41, 42, 43, 45, 51, 52, 53, 54\}$$

(b) What is the probability that the first number is even? (5 marks)

Let $A =$ 'first number even'. Then

$$\begin{aligned} A &= \{21, 23, 24, 25, 41, 42, 43, 45\} \\ P(A) &= \frac{8}{20} = \frac{2}{5}. \end{aligned}$$

(c) What is the probability that the second number is greater than the first? (5 marks)

Let $B =$ 'second number greater than first'. Then

$$\begin{aligned} B &= \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\} \\ P(B) &= \frac{10}{20} = \frac{1}{2}. \end{aligned}$$

3 (15 marks) You are given that a coin has probability p of showing heads when it is tossed, where $p > 0$. You toss the coin three times.

(a) What is the probability that you obtain three heads? (5 marks)

Since different tosses are independent,

$$P(\{HHH\}) = p \times p \times p = p^3.$$

(b) What is the probability that you obtain two heads and one tail (in any order)? (5 marks)

The probability of tails is $1 - p$. So

$$P(\{HHT, HTH, THH\}) = 3p^2(1 - p).$$

(c) Find the value of p for which the probability of three heads is equal to the probability of two heads and one tail. (5 marks)

This occurs when

$$\begin{aligned} p^3 &= 3p^2(1-p), \\ p &= 3(1-p), \qquad \text{(since } p \neq 0\text{)} \\ 4p &= 3, \\ p &= \frac{3}{4}. \end{aligned}$$

4 (25 marks)

(a) What is meant by the *conditional probability* of event A given event B? (5 marks)

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

if $P(B) \neq 0$.

(b) What does it mean to say that the events A and B are *independent*? (5 marks)

$$P(A \cap B) = P(A)P(B).$$

(c) Prove that, if A and B are independent, then

$$P(A \mid B) = P(A).$$

(15 marks)

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \qquad \text{(by definition)} \\ &= \frac{P(A)P(B)}{P(B)} \qquad \text{(by independence of } A \text{ and } B\text{)} \\ &= P(A). \end{aligned}$$

5 (15 marks) A discrete random variable V has the following probability mass function.

x	3	6
$P(V = x)$	$\frac{1}{3}$	$\frac{2}{3}$

(a) Find $E(V)$. (5 marks)

$$E(V) = 3 \times \frac{1}{3} + 6 \times \frac{2}{3} = 5.$$

(b) Find $\mathrm{Var}(V)$. (10 marks)

$$E(V^2) = 3^2 \times \frac{1}{3} + 6^2 \times \frac{2}{3} = 27,$$

so

$$\mathrm{Var}(V) = E(V^2) - E(V)^2 = 27 - 5^2 = 2.$$

6 (15 marks) In a certain university, 60% of the students play football, 40% play cricket, 30% swim, 20% play both football and cricket, 15% play football and swim, 5% play cricket and swim, and 1% swim and play both football and cricket. A student is chosen at random in that university. What is the probability that the chosen student plays football or plays cricket or swims?

Let $F =$ "student plays football", $C =$ "student plays cricket", and $S =$ "student swims". Then

$$\begin{gathered} P(F) = 0.6, \qquad P(C) = 0.4, \qquad P(S) = 0.3, \\ P(F \cap C) = 0.2, \qquad P(F \cap S) = 0.15, \qquad P(C \cap S) = 0.05, \\ P(F \cap C \cap S) = 0.01. \end{gathered}$$

By the Inclusion-Exclusion Principle,

$$\begin{aligned} P(F \cup C \cup S) &= P(F) + P(C) + P(S) - P(F \cap C) - P(F \cap S) - P(C \cap S) \\ &\quad + P(F \cap C \cap S) \\ &= 0.6 + 0.4 + 0.3 - 0.2 - 0.15 - 0.05 + 0.01 \\ &= 0.91. \end{aligned}$$