

**What is it?**

Let  $a_1, a_2, a_3, \dots$  be numbers. The notation

$$\sum_{i=1}^n a_i$$

(read “sum, from  $i$  equals 1 to  $n$ , of  $a_i$ ”), means: add up the numbers  $a_1, a_2, \dots, a_n$ ; that is,

$$\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n.$$

The notation  $\sum_{j=1}^n a_j$  means exactly the same thing. The variable  $i$  or  $j$  is called a “dummy variable”.

The notation  $\sum_{i=1}^m a_i$  is not the same, since (if  $m$  and  $n$  are different) it is telling us to add up a different number of terms.

The sum doesn’t have to start at 1. For example,

$$\sum_{i=10}^{20} a_i = a_{10} + a_{11} + \dots + a_{20}.$$

Sometimes I get lazy and don’t bother to write out the values: I just say  $\sum_i a_i$  to mean: add up all the relevant values. For example, if  $X$  is a discrete random variable, then we say that

$$E(X) = \sum_i a_i P(X = a_i)$$

where the sum is over all  $i$  such that  $a_i$  is a value of the random variable  $X$ .

**Typographical note:**  $\sum_{i=1}^n a_i$  is exactly the same as  $\sum_{i=1}^n a_i$ .

## Manipulation

The following four rules hold.

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i. \quad (1)$$

Imagine the  $a$ s and  $b$ s written out with  $a_1 + b_1$  on the first line,  $a_2 + b_2$  on the second line, and so on. The left-hand side says: add the two terms in each line, and then add up all the results. The right-hand side says: add the first column (all the  $a$ s) and the second column (all the  $b$ s), and then add the results. The answers must be the same.

$$\left( \sum_{i=1}^n a_i \right) \cdot \left( \sum_{j=1}^m b_j \right) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j. \quad (2)$$

The double sum says add up all these products, for all values of  $i$  and  $j$ . A simple example shows how it works:  $(a_1 + a_2)(b_1 + b_2) = a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$ .

If  $c$  is a constant then

$$\sum_{i=1}^n c a_i = c \sum_{i=1}^n a_i. \quad (3)$$

The left-hand side says multiply every term by  $c$  and then add up the results; the right-hand side says add up the terms and then multiply the total by  $c$ . Because multiplication is distributive over addition, the two sides must be the same.

If in place of numbers, we have functions of  $x$ , then we can “differentiate term-by-term”:

$$\frac{d}{dx} \sum_{i=1}^n f_i(x) = \sum_{i=1}^n \frac{d}{dx} f_i(x). \quad (4)$$

The left-hand side says: add up the functions and differentiate the sum. The right says: differentiate each function and add up the derivatives.

## Infinite sums

Sometimes we meet infinite sums, which we write as  $\sum_{i=1}^{\infty} a_i$  for example. This doesn't just mean "add up infinitely many values", since that is not possible. We need Analysis to give us a definition in general. But sometimes we know the answer another way: for example, if  $a_i = ar^{i-1}$ , where  $-1 < r < 1$ , then

$$\sum_{i=1}^{\infty} a_i = a + ar + ar^2 + \cdots = \frac{a}{1-r},$$

using the formula for the sum of the "geometric series". You also need to know the sum of the "exponential series"

$$\sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots = e^x.$$

Do the four rules of the preceding section hold? Sometimes yes, sometimes no. In Analysis you will see some answers to this question.

In all the examples you meet in Probability I, the rules will be valid.