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MAS 108 Probability I

Solutions 5 Autumn 2005

Here are possible solutions to Questions 2 and 3 on Assignment 5. If your method of solution is different it may still be correct. Come and ask me if you are not sure.

2 Let $G =$ "has one C gene" and let $T =$ "test is positive".
By the Theorem of Total Probability,

$$P(T) = P(G) \times P(T \mid G) + P(G') \times P(T \mid G') = \frac{2}{51} \times \frac{95}{100} + \frac{49}{51} \times \frac{2}{100} = \frac{288}{5100}.$$

$$P(T') = 1 - P(T) = \frac{4812}{5100}.$$

(a) By Bayes' Theorem, $P(G \mid T) = \dfrac{P(T \mid G) \times P(G)}{P(T)} = \dfrac{\frac{95}{100} \times \frac{2}{51}}{\frac{288}{5100}} = \dfrac{190}{288} \approx 0.6597.$

(b) By Bayes' Theorem, $P(G' \mid T') = \dfrac{P(T' \mid G') \times P(G')}{P(T')} = \dfrac{\frac{98}{100} \times \frac{49}{51}}{\frac{4812}{5100}} = \dfrac{4802}{4812} \approx 0.9979.$

TURN OVER

3 [The trick is to denote all the relevant events by letters.]

Let $A =$ "passes no exams at first resits", $B =$ "passes exactly one exam at first resits", $Q_1 =$ "qualifies for second year after first resits", $Q_2 =$ "qualifies for second year after second resits", and $Q =$ "ever qualifies for second year ".

(a) To qualify for the second year he needs to pass two more exams, so $Q_1' = A \cup B$ and $P(Q_1) = 1 - P(A) - P(B)$. But $P(A) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$ and $P(B) = 4 \times \left(\frac{1}{2}\right)^4 = \frac{4}{16}$ so $P(Q_1') = \frac{5}{11}$ and $P(Q_1) = \frac{11}{16}$.

(b) If he passes no exams at the first resits then he is in the same position at the second resits, so $P(Q_2 \mid A) = P(Q_1) = \frac{11}{16}$. If he passes one exam at the first resits then he takes three exams at the second resits and he will qualify unless he fails all three. So $P(Q_2 \mid B) = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$.

Now, Q_1' is the disjoint union of A and B, so $Q_2 \cap Q_1'$ is the disjoint union of $Q_2 \cap A$ and $Q_2 \cap B$, so

$$\begin{aligned} P(Q_2 \cap Q_1') &= P(Q_2 \cap A) + P(Q_2 \cap B) \\ &= P(A) \times P(Q_2 \mid A) + P(B) \times P(Q_2 \mid B) \\ &= \frac{1}{16} \times \frac{11}{16} + \frac{4}{16} \times \frac{7}{8} = \frac{67}{16^2}. \end{aligned}$$

Then $P(Q_2 \mid Q_1') = \dfrac{P(Q_2 \cap Q_1')}{P(Q_1')} = \dfrac{67}{16^2} \times \dfrac{16}{5} = \dfrac{67}{80}$.

(c) By the Theorem of Total Probability, $P(Q) = P(Q_1) \times P(Q \mid Q_1) + P(Q_1') \times P(Q \mid Q_1')$. But $P(Q \mid Q_1) = 1$ [because $Q \subseteq Q_1$] and $P(Q \mid Q_1') = P(Q_2 \mid Q_1') = 67/80$ [because $Q \cap Q_1' = Q_2 \cap Q_1'$]. So

$$P(Q) = \frac{11}{16} \times 1 + \frac{5}{16} \times \frac{67}{80} = \frac{243}{256}.$$

[*Or* you could say that Q is the disjoint union of Q_1, $Q_2 \cap A$ and $Q_2 \cap B$.]

(d) We are being asked for $P(A \mid Q)$. [Often this is the hardest part, working out exactly which conditional probability is needed.]

By Bayes' Theorem,

$$P(A \mid Q) = \frac{P(Q \mid A) \times P(A)}{P(Q)} = \frac{P(Q_2 \mid A) \times P(A)}{P(Q)} = \frac{11}{16} \times \frac{1}{16} \times \frac{256}{243} = \frac{11}{243}.$$