MAS 108 Probability I

Notes 3 Autumn 2005

Independence

Two events A and B are said to be *independent* if

$$P(A \cap B) = P(A) \times P(B).$$

This is the definition of independence of events. If you are asked in an exam to define independence of events, this is the correct answer. Do not say that two events are independent if one has no influence on the other; and *under no circumstances* say that A and B are independent if $A \cap B = \emptyset$ (this is the statement that A and B are disjoint, which is quite a different thing!) Also, do not ever say that $P(A \cap B) = P(A) \times P(B)$ unless you have some good reason for assuming that A and B are independent (see below).

In general, *don't assume that two events are independent* unless either

(a) they depend on *different* tosses of a coin or rolls of a die; or

(b) you are told in the question to assume this!

Rather, you should calculate probabilities to see whether or not they are independent.

In general, it is always OK to assume that the outcomes of different tosses of a coin, or different throws of a die, are independent. This holds even if the probabilities are not all equally likely. We will see an example later.

Example If we toss a coin more than once, or roll a die more than once, then you may assume that different tosses or rolls are independent. More precisely, if we roll a fair six-sided die twice, then the probability of getting 4 on the first throw and 5 on the second is $1/36$, since we assume that all 36 combinations of the two throws are equally likely. But $(1/36) = (1/6) \cdot (1/6)$, and the separate probabilities of getting 4 on the first throw and of getting 5 on the second are both equal to $1/6$. So the two events are independent. This would work just as well for any other combination.

Example (a) I roll a fair 6-sided die. Let A be the event that the number is 3 or smaller, and B the event that it is even. Then $A = \{1,2,3\}$, $B = \{2,4,6\}$, $A \cap B = \{2\}$; so $P(A) = 1/2$, $P(B) = 1/2$, and $P(A \cap B) = 1/6$. So the events are not independent.

(b) I roll a fair 6-sided die twice. Let A be the event that the number on the first roll is 3 or smaller, and B the event that the number on the second roll is even. These events should be independent, since they depend on different rolls. Let us see. We have:

$$\begin{aligned} A &= \{(1,1),\ldots,(1,6),(2,1),\ldots,(2,6),(3,1),\ldots,(3,6)\}, \\ B &= \{(1,2),\ldots,(6,2),(1,4),\ldots,(6,4),(1,6),\ldots,(6,6)\}, \\ A \cap B &= \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6), \end{aligned}$$

$$P(A) = 18/36 = 1/2, \quad P(B) = 18/36 = 1/2, \quad P(A \cap B) = 9/36 = 1/4.$$

So the events are independent.

Example I have four pens in my satchel; they are red, green, blue, and purple. I choose two pens with replacement. Let A be the event that the first pen is red or green, and B the event that the second pen is red or green. Are A and B independent? The sample space is

$$\begin{array}{cccc} \{RR, & RG, & RB, & RP, \\ GR, & GG, & GB, & GP, \\ BR, & BG, & BB, & BP, \\ PR, & PG, & PB, & PP\}, \end{array}$$

and

$$\begin{aligned} A &= \{RR,\ RG,\ RB,\ RP,\ GR,\ GG,\ GB,\ GP\} \\ B &= \{RR,\ RG, GR,\ GG,\ BR,\ BG,\ PR,\ PG\} \\ A \cap B &= \{RR,\ RG,\ GR,\ GG\}. \end{aligned}$$

Thus $P(A) = 1/2$, $B(B) = 1/2$ and $P(A \cap B) = 1/4 = P(A) \times P(B)$ and so the events are independent.

On the other hand, suppose that I sample *without* replacement. Then

$$\begin{aligned} \mathcal{S} &= \{RG,\ RB,\ RP,\ GR,\ GB,\ GP,\ BR,\ BG,\ BP,\ PR,\ PG,\ PB\} \\ A &= \{RG,\ RB,\ RP,\ GR,\ GB,\ GP\} \\ B &= \{RG, GR,\ BR,\ BG,\ PR,\ PG\} \\ A \cap B &= \{RG,\ GR\}. \end{aligned}$$

Now we have $P(A) = P(B) = 1/2$ but $P(A \cap B) = 1/6 \neq P(A) \times P(B)$ so the events are not independent.

Example Two fair ten-sided dice are thrown, independently of each other, so that

$$\mathcal{S} = \{(i, j) : 1 \leq i \leq 10,\ 1 \leq j \leq 10\}$$

and every outcome has probability $1/100$. Let A be 'the first is 10', B be 'doubles' and C be 'at least one is an odd number'. Is it obvious that A and B should be independent of each other? What about B and C?

Now, $P(A) = P(B) = 1/10$. The complement C' of C is 'both are even', so $P(C') = 1/4$ and therefore $P(C) = 3/4$.

$$\begin{aligned} A \cap B &= \{(10, 10)\} \\ B \cap C &= \{(i, i) : i = 1,\ 3,\ 5,\ 7,\ 9\}, \end{aligned}$$

so $P(A \cap B) = 1/100 = P(A) \times P(B)$ and A and B are independent. On the other hand, $P(B \cap C) = 5/100 = 1/20 \neq P(B) \times P(C)$ so B and C are not independent of each other.

Mutual independence

This section is a bit technical. You will need to know the conclusions.

Suppose that A, B and C are events. If all three pairs of events happen to be independent, can we then conclude that $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$? At first sight this seems very reasonable; in Axiom 3, we only required all pairs of events to be exclusive in order to justify our conclusion. Unfortunately it is not true ...

Example In the example with two ten-sided dice, let D be 'the second is 10'. You can check that A and D are independent of each other, and that B and D are independent of each other. We already know that A and B are independent of each other. However, $A \cap B = A \cap D = B \cap D$ so we know that if any two of these events occur then the third must too. Here $P(A \cap B \cap D) = 1/100 \neq P(A) \times P(B) \times P(D)$.

Thus, the definition of mutual independence for three events must require more than just the independence of each pair. We say that three events A, B, C are *mutually independent* if

- each pair of events is independent;
- $P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$.

So, in the ten-sided dice example, A, B and D are not mutually independent.

How do we extend this to any number of events? The main difficulty is in finding a good notation! The correct definition runs as follows.

Let $A_1, \ldots, A_n$ be events. We say that these events are *mutually independent* if, for every t with $2 \le t \le n$ and all indices $i_1, i_2, \ldots, i_t$ with $1 \le i_1 < i_2 < \cdots < i_t \le n$, we have

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_t}) = P(A_{i_1}) \times P(A_{i_2}) \times \cdots \times P(A_{i_t}).$$

Sometimes, instead of saying "the events $A_1, \ldots, A_n$ are mutually independent", we say "each event is independent of all the others". This form of words has exactly the same meaning.

You should not assume in general that events are mutually independent. You can only assume this if *either*

(a) they depend on *different* tosses of a coin or rolls of a die; or

(b) you are told in the question to assume this!

For example, if I toss a coin six times, the three events 'same result on tosses 1 and 2', 'more heads than tails on tosses 3, 4 and 5', and 'heads on toss 6' are mutually independent.

Example A coin has probability p of coming down heads, and probability q of coming down tails, where $q = 1 - p$. It is tossed three times, independently.

$$\begin{aligned} P(HTT) &= P(\text{1st is H}) \times P(\text{2nd is T}) \times P(\text{3rd is T}), \qquad \text{by independence}, \\ &= p \times q \times q = pq^2. \end{aligned}$$

Similarly, $P(THT) = P(TTH) = pq^2$, so $P(\text{1 head and 2 tails in any order}) = 3pq^2$. In general, if the coin is tossed n times, the probability that it comes down heads exactly r times is ${}^n\mathrm{C}_r p^r q^{n-r}$.

For technical reasons, we do not usually define A and B to be independent if either of them has probability zero. You need not worry about this.

Properties of independence

Proposition If A and B are independent, then A and B' are independent.

I didn't prove this in lectures. Can you prove it?

Corollary If A and B are independent, so are A' and B'.

Apply the Proposition twice, first to A and B (to show that A and B' are independent), and then to B' and A (to show that B' and A' are independent).

More generally, if events $A_1, \ldots, A_n$ are mutually independent, and we replace some of them by their complements, then the resulting events are mutually independent. We have to be a bit careful though. For example, A and A' are not usually independent!

Results like the following are also true, though we don't stop to prove this.

Proposition Let events A, B, C be mutually independent. Then A and $B \cap C$ are independent, and A and $B \cup C$ are independent.

Stopping rules

Often an experiment consists of performing some action repeatedly until some condition is met.

Suppose that Carole takes a driving test. She is allowed to keep taking the test until she passes. Of course, if she passes the test, she doesn't need to take it again. So the sample space is

$$\mathcal{S} = \{P,\ FP,\ FFP,\ FFFP\ \ldots\},$$

where, for example, FFP denotes the outcome that she fails twice and passes on her third attempt. The sample space is infinite.

In Mathematics examinations we are not so liberal. You are allowed to take the exam up to three times but no more. Now the sample space is

$$\mathcal{S} = \{P,\ FP,\ FFP,\ FFF\}.$$

If all outcomes were equally likely, then your chance of eventually passing the exam would be $3/4$.

But it is unreasonable here to assume that all the outcomes are equally likely. For example, you may be very likely to pass on the first attempt. Let us assume that the probability that you pass the exam is 0.8. (By Proposition 1, your chance of failing is 0.2.) Let us further assume that, no matter how many times you have failed, your chance of passing at the next attempt is still 0.8; in other words, that the different attempts are independent of each other. Then we have

$$\begin{aligned} P(P) &= 0.8, \\ P(FP) &= 0.2 \times 0.8 = 0.16, \\ P(FFP) &= 0.2^2 \times 0.8 = 0.032, \\ P(FFF) &= 0.2^3 = 0.008. \end{aligned}$$

Thus the probability that you eventually pass is $P(\{P,\ FP,\ FFP\}) = 0.8 + 0.16 + 0.032 = 0.992$. Alternatively, you eventually pass *unless* you fail three times, so the probability is $1 - 0.008 = 0.992$.

A *stopping rule* is a rule of the type described here, namely, continue the experiment until some specified occurrence happens.

The simplest kind of stopping rule is 'perform the experiment a fixed number of times'. For example, if I toss a fair coin three times, independently, the sample space is

$$\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

If the coin is fair, then independence shows that all outcomes are equally likely, with probability $1/8$.

The experiment may potentially be infinite.

For example, if you toss a coin repeatedly until you obtain heads, the sample space is

$$\mathcal{S} = \{H, TH, TTH, TTTH, \ldots\}$$

since in principle you may get arbitrarily large numbers of tails before the first head. (We have to allow all possible outcomes.)

In the Mathematics exam, the rule is 'stop if either you pass or you have taken the exam three times'. This ensures that the sample space is finite. The analogous thing for coins would be 'stop when you get heads, or when you have tossed three times', and the sample space is

$$\mathcal{S} = \{H, TH, TTH, TTT\}.$$

The outcomes are not equally likely; we would have $P(H) = 1/2$, $P(TTH) = 1/4$, $P(TTH) = P(TTT) = 1/8$.

Another rule is 'best of three'. If you toss a coin with this rule, then if two heads (or two tails) come up, the experiment can stop since the result is decided. The sample space is

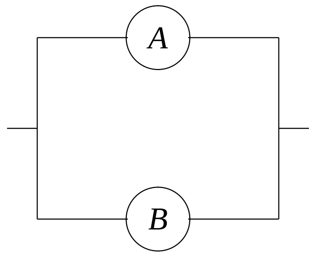
$$\mathcal{S} = \{HH, HTH, HTT, THH, THT, TT\},$$

and $P(HH) = P(TT) = 1/4$ while the other outcomes all have probability $1/8$.

Networks and reliability

Often we can assume that different electrical components, or different pipes in a water system, or different parts of a transport network, behave independently. Here are some examples.

Example Connections in parallel



Two electrical components are connected in parallel. Current flows so long as at least one component is working. The probability that component A fails is $1/10$ and the probability that B fails is $1/20$, independently of A. What is the probability that current flows?

At risk of some confusion, we use the *letters* A and B for the *events* 'component A works' and 'component B works' respectively. Let Y be the event that current flows. Now, current flows if *either* A is working *or* B is working, so $Y = A \cup B$. Therefore

$$\begin{aligned} P(Y) = P(A \cup B) &= P(A) + P(B) - P(A \cap B), && \text{by inclusion-exclusion,} \\ &= P(A) + P(B) - P(A) \times P(B), && \text{because } A \text{ and } B \text{ are independent,} \\ &= \frac{9}{10} + \frac{19}{20} - \frac{9}{10} \times \frac{19}{20} = \frac{199}{200}. \end{aligned}$$

Alternatively, we could argue that $Y' = A' \cap B'$. Then independence of A' and B' gives $P(Y') = P(A') \times P(B') = (1 - P(A)) \times (1 - P(B)) = 1/200$, so $P(Y) = 199/200$.

There is often more than one way to tackle such a problem. The important thing is to be clear about where you are using independence and where you are using other rules such as inclusion-exclusion.

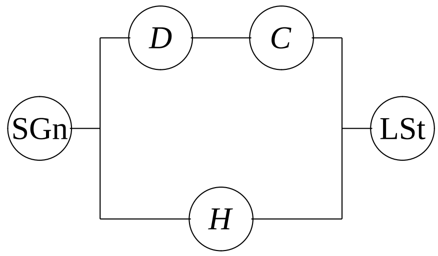
Example Connections in series

When electrical components are connected in series, current flows only if both are working. Suppose that A and B work independently, as above.

$$\begin{aligned} P(Y) = P(A \cap B) &= P(A) \times P(B), && \text{by independence,} \\ &= \frac{9}{10} \times \frac{19}{20} = \frac{171}{200}. \end{aligned}$$

Example From Assignment 3

You are at Stepney Green station and you want to get to Liverpool Street station. You can either go directly by the Hammersmith and City line, or you can take the District line to Mile End followed by the Central line to Liverpool Street. For simplicity, let us assume that there are no other Underground lines.



The probability that the Hammersmith and City line is working is 0.9; the probability that the District line is working is 0.8; and the probability that the Central line is working is 0.75. Assume that each line is independent of the others. What is the probability that you can get to Liverpool Street by Underground train?

Write D for the event 'the District line is working', and so on. Then the event that we want is $(D \cap C) \cup H$. Now

$$\begin{aligned} P((D \cap C) \cup H)) &= P(D \cap C) + P(H) - P(D \cap C \cap H) \\ &\qquad \text{(by Inclusion-Exclusion)} \\ &= P(D) \times P(C) + P(H) - P(D) \times P(C) \times P(H) \\ &\qquad \text{(by mutual independence)} \\ &= (0.8) \times (0.75) + (0.9) - (0.8) \times (0.75) \times (0.9) \\ &= 0.96. \end{aligned}$$

There is a trap here which you should take care to avoid. You might be tempted to say

$$(D \cap C) \cup H = (D \cup H) \cap (C \cup H),$$

by the distributive law; then calculate using inclusion-exclusion and independence that

$$\begin{aligned} P(D \cup H) = P(D) + P(H) - P(D)P(H) &= 0.8 + 0.9 - (0.8) \times (0.9) = 0.98, \\ P(C \cup H) = P(C) + P(H) - P(C)P(H) &= 0.75 + 0.9 - (0.75) \times (0.9) = 0.975, \end{aligned}$$

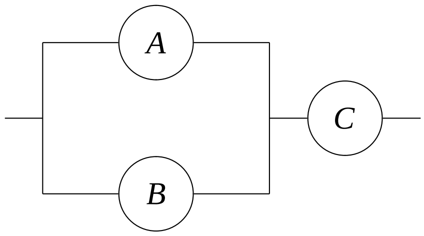
and finally conclude, using independence, that

$$P((D \cup H) \cap (C \cup H)) = (0.98) \times (0.975) = 0.9555.$$

This is correct right up to the last step, where we have been sloppy about using independence. It does *not* follow from what we are given that $D \cup H$ and $C \cup H$ are independent!

Example Based on a question from the 1999 exam.

Water flows from left to right in the pipework shown in the diagram so long as it can find an unblocked route from left to right. Blockages occur independently in the three named pipes. The probability that pipe A is blocked is $1/4$; the probability that pipe B is blocked is $2/5$; and the probability that pipe C is blocked is $1/6$.



Find the probability that water flows.

Let A denote the event that pipe A is *unblocked*, and similarly for B and C. Then $P(A) = 3/4$, $P(B) = 3/5$ and $P(C) = 5/6$. Now

$$\begin{aligned}
P(\text{water flows}) &= P[(A \cup B) \cap C] \\
&= P(A \cup B) \times P(C), \\
&\qquad \text{because } A \cup B \text{ is independent of } C, \\
&= [P(A) + P(B) - P(A \cap B)] \times P(C), \\
&\qquad \text{by inclusion-exclusion}, \\
&= [P(A) + P(B) - P(A)P(B)] \times P(C), \\
&\qquad \text{because } A \text{ is independent of } B, \\
&= \left(\frac{3}{4} + \frac{3}{5} - \frac{9}{20}\right) \times \left(\frac{5}{6}\right) \\
&= \frac{18}{20} \times \frac{5}{6} = \frac{3}{4}.
\end{aligned}$$